

TWISTING OF SIEGEL PARAMODULAR FORMS

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ABSTRACT. Let $S_k(\Gamma^{\text{para}}(N))$ be the space of Siegel paramodular forms of level N and weight k . Let $p \nmid N$ and let χ be a nontrivial quadratic Dirichlet character mod p . Based on [JR2], we define a linear twisting map $\mathcal{T}_\chi : S_k(\Gamma^{\text{para}}(N)) \rightarrow S_k(\Gamma^{\text{para}}(Np^4))$. We calculate an explicit expression for this twist and give the commutation relations of this map with the Hecke operators and Atkin-Lehner involution for primes $\ell \neq p$.

1. INTRODUCTION

Let k and N be positive integers and let p be a prime with $p \nmid N$. Let $S_k(\Gamma_0(N))$ denote the space of elliptic modular cusp forms of weight k with respect to $\Gamma_0(N) \subset \text{SL}(2, \mathbb{Z})$ and let χ be a nontrivial quadratic Dirichlet character mod p . There is a natural twisting map $\mathcal{T}_\chi : S_k(\Gamma_0(N)) \rightarrow S_k(\Gamma_0(Np^2))$ such that if $f \in S_k(\Gamma_0(N))$ then

$$\mathcal{T}_\chi(f) = \sum_{u \in (\mathbb{Z}/p\mathbb{Z})^\times} \chi(u) f|_k \begin{bmatrix} 1 & u/p \\ & 1 \end{bmatrix}. \quad (1)$$

Moreover, the Fourier expansion of the twist $\mathcal{T}_\chi(f)$ is given by

$$\mathcal{T}_\chi(f)(z) = \sum_{n=1}^{\infty} W(\chi) \chi(n) a_n e^{2\pi i n z}, \quad (2)$$

where $W(\chi) = \sum_{u \in (\mathbb{Z}/p\mathbb{Z})^\times} \chi(u) e^{2\pi i u/p}$ is the Gauss sum of χ and $f(z) = \sum_{n=1}^{\infty} a_n e^{2\pi i n z}$ is the Fourier expansion of f . See, for example, [Sh] Proposition 3.64. A calculation verifies that for a prime $\ell \neq p$,

$$T(\ell) \mathcal{T}_\chi = \chi(\ell) \mathcal{T}_\chi T(\ell) \quad \text{and} \quad (\mathcal{T}_\chi f)|_k W_\ell = \chi(\ell)^{\text{val}_\ell(N)} \mathcal{T}_\chi(f|_k W_\ell), \quad (3)$$

where $f \in S_k(\Gamma_0(N))$, $T(\ell)$ is the Hecke operator, and W_ℓ is the Atkin-Lehner involution at ℓ as defined in Section 2.4 of [Cr], for example. By identifying $S_k(\Gamma_0(N))$ with automorphic forms on the adèles of $\text{GL}(2)$, it is evident that this twisting only acts on the automorphic form at the prime p . Hence the global twist is induced by a local twisting map for representations of $\text{GL}(2, \mathbb{Q}_p)$. In our previous work, [JR2], we constructed an analogous local twisting map for paramodular representations of $\text{GSp}(4, \mathbb{Q}_p)$ with trivial central character.

In this paper, we investigate the twisting map, \mathcal{T}_χ , on the space of Siegel paramodular forms induced by the local map from [JR2]. Our main result, Theorem 3.1, proves a formula for \mathcal{T}_χ similar to, but more involved than, the formula (1) and proves commutation relations for the paramodular Hecke operators and Atkin-Lehner involution analogous to (3). Moreover, it follows from Theorem 1.2 of [JR2] that the map \mathcal{T}_χ is not zero in general. Hence, our theorem may provide a source of examples to study conjectures such as the paramodular conjecture [BK] and the paramodular Böcherer's conjecture [RT]. Finally, the formula (5) in Theorem 3.1 enables the computation of

the Fourier coefficients of the twisted paramodular form $\mathcal{T}_\chi(F)$, resulting in a formula analogous to (2). This will appear in a subsequent paper.

The paper is organized as follows. Section 2 introduces some necessary definitions and notation. In Section 3, we present and prove the main theorem. However, the proof relies on lengthy local calculations which are presented in Section 4 and the appendix [JR-a]. Note that in Section 4, $\mathrm{GSp}(4)$ is defined with respect to the symplectic form used in [RS] rather than the one given in Section 2. Finally in Section 5, the local results from Section 4 are translated to the setting of Section 3.

2. NOTATION

Let M be a positive integer and let $\chi : (\mathbb{Z}/M\mathbb{Z})^\times \rightarrow \mathbb{C}^\times$ be a Dirichlet character. We let \mathbb{A} denote the adeles of \mathbb{Q} and define an associated Hecke character $\mathbb{Q}^\times \backslash \mathbb{A}^\times \rightarrow \mathbb{C}^\times$, denoted by χ , as follows. Recall that $\mathbb{A}^\times = \mathbb{Q}^\times \mathbb{R}_{>0}^\times \prod_{\ell < \infty} \mathbb{Z}_\ell^\times$, with the groups embedded in \mathbb{A}^\times in the usual ways. In fact, the map defined by $(q, r, n) \mapsto qrn$ defines an isomorphism of topological groups

$$\mathbb{Q}^\times \times \mathbb{R}_{>0}^\times \times \prod_{\ell < \infty} \mathbb{Z}_\ell^\times \xrightarrow{\sim} \mathbb{A}^\times.$$

Let $M = \ell_1^{\mathrm{val}_{\ell_1}(M)} \cdots \ell_t^{\mathrm{val}_{\ell_t}(M)}$ be the prime factorization of M . We consider the composition

$$\begin{aligned} \mathbb{Z}_{\ell_1}^\times \times \cdots \times \mathbb{Z}_{\ell_t}^\times &\longrightarrow \mathbb{Z}_{\ell_1}^\times / (1 + \ell_1^{\mathrm{val}_{\ell_1}(M)} \mathbb{Z}_{\ell_1}) \times \cdots \times \mathbb{Z}_{\ell_t}^\times / (1 + \ell_t^{\mathrm{val}_{\ell_t}(M)} \mathbb{Z}_{\ell_t}) \\ &\xrightarrow{\sim} (\mathbb{Z}/\ell_1^{\mathrm{val}_{\ell_1}(M)} \mathbb{Z})^\times \times \cdots \times (\mathbb{Z}/\ell_t^{\mathrm{val}_{\ell_t}(M)} \mathbb{Z})^\times \xrightarrow{\sim} (\mathbb{Z}/M\mathbb{Z})^\times \xrightarrow{\chi} \mathbb{C}^\times. \end{aligned}$$

We denote the restriction of this composition to $\mathbb{Z}_{\ell_i}^\times$ by χ_{ℓ_i} . If $p \nmid M$, then we define $\chi_\ell : \mathbb{Z}_\ell^\times \rightarrow \mathbb{C}^\times$ to be the trivial character. For each finite prime p , χ_ℓ is a continuous character of \mathbb{Z}_ℓ^\times , and $\chi_\ell(1 + \ell^{\mathrm{val}_\ell(M)} \mathbb{Z}_\ell) = 1$. We define the corresponding Hecke character $\chi : \mathbb{Q}^\times \backslash \mathbb{A}^\times \rightarrow \mathbb{C}^\times$ as the composition

$$\mathbb{A}^\times \xrightarrow{\sim} \mathbb{Q}^\times \times \mathbb{R}_{>0}^\times \times \prod_{\ell < \infty} \mathbb{Z}_\ell^\times \xrightarrow{\mathrm{proj}} \prod_{\ell < \infty} \mathbb{Z}_\ell^\times \xrightarrow{\prod \chi_\ell} \mathbb{C}^\times. \quad (4)$$

We see that if $a \in \mathbb{Z}$ with $(a, M) = 1$, then

$$\chi(a) = \chi_{\ell_1}(a) \cdots \chi_{\ell_t}(a).$$

Let

$$J = \begin{bmatrix} & & & \mathbf{1}_2 \\ & & & \\ & & & \\ -\mathbf{1}_2 & & & \end{bmatrix}.$$

We define the algebraic \mathbb{Q} -group $\mathrm{GSp}(4)$ as the set of all $g \in \mathrm{GL}(4)$ such that ${}^t g J g = \lambda(g) J$ for some $\lambda(g) \in \mathrm{GL}(1)$ called the multiplier of g . Let $\mathrm{GSp}(4, \mathbb{R})^+$ be the subgroup of $g \in \mathrm{GSp}(4, \mathbb{R})$ such that $\lambda(g) > 0$. The kernel of $\lambda : \mathrm{GSp}(4) \rightarrow \mathrm{GL}(1)$ is the symplectic group $\mathrm{Sp}(4)$. Let N and k be positive integers. We define the paramodular group of level N to be

$$\Gamma^{\mathrm{para}}(N) = \mathrm{Sp}(4, \mathbb{Q}) \cap \begin{bmatrix} \mathbb{Z} & \mathbb{Z} & N^{-1}\mathbb{Z} & \mathbb{Z} \\ N\mathbb{Z} & \mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ N\mathbb{Z} & N\mathbb{Z} & \mathbb{Z} & N\mathbb{Z} \\ N\mathbb{Z} & \mathbb{Z} & \mathbb{Z} & \mathbb{Z} \end{bmatrix}.$$

We also define local paramodular groups. For ℓ a prime of \mathbb{Q} and r a non-negative integer, let $K^{\mathrm{para}}(\ell^r)$ be the paramodular subgroup of $\mathrm{GSp}(4, \mathbb{Q}_\ell)$ of level ℓ^r , i.e., the subgroup of elements

$k \in \mathrm{GSp}(4, \mathbb{Q}_\ell)$ such that $\lambda(k) \in \mathbb{Z}_\ell^\times$ and

$$k \in \begin{bmatrix} \mathbb{Z}_\ell & \mathbb{Z}_\ell & \ell^{-r}\mathbb{Z}_\ell & \mathbb{Z}_\ell \\ \ell^r\mathbb{Z}_\ell & \mathbb{Z}_\ell & \mathbb{Z}_\ell & \mathbb{Z}_\ell \\ \ell^r\mathbb{Z}_\ell & \ell^r\mathbb{Z}_\ell & \mathbb{Z}_\ell & \ell^r\mathbb{Z}_\ell \\ \ell^r\mathbb{Z}_\ell & \mathbb{Z}_\ell & \mathbb{Z}_\ell & \mathbb{Z}_\ell \end{bmatrix}.$$

Note that

$$\Gamma^{\mathrm{para}}(N) = \mathrm{GSp}(4, \mathbb{Q}) \cap \mathrm{GSp}(4, \mathbb{R})^+ \prod_{\ell < \infty} K^{\mathrm{para}}(\ell^{\mathrm{val}_\ell(N)}),$$

with intersection in $\mathrm{GSp}(4, \mathbb{A})$.

For n a positive integer, let \mathfrak{H}_n denote the Siegel upper half plane of degree n with $I = i\mathbf{1}_{2n}$. The group $\mathrm{GSp}(4, \mathbb{R})^+$ acts on \mathfrak{H}_2 via

$$h\langle Z \rangle = (AZ + B)(CZ + D)^{-1}, \quad h = \begin{bmatrix} A & B \\ C & D \end{bmatrix}, \quad Z \in \mathfrak{H}_2.$$

Denote the factor of automorphy by $j(h, Z) = \det(CZ + D)$. If $F : \mathfrak{H}_2 \rightarrow \mathbb{C}$ is a function and $h \in \mathrm{GSp}(4, \mathbb{R})^+$, then we define $F|_k h : \mathfrak{H}_2 \rightarrow \mathbb{C}$ by

$$(F|_k h)(Z) = \lambda(h)^k j(h, Z)^{-k} F(h\langle Z \rangle), \quad Z \in \mathfrak{H}_2.$$

Let $\Gamma \subset \mathrm{GSp}(4, \mathbb{Q})$ be a group commensurable with $\mathrm{Sp}(4, \mathbb{Z})$. We define $S_k(\Gamma)$ to be the complex vector space of functions $F : \mathfrak{H}_2 \rightarrow \mathbb{C}$ such that

- (1) F is holomorphic;
- (2) $F|_k \gamma = F$ for all $\gamma \in \Gamma$;
- (3) $\lim_{t \rightarrow \infty} (F|_k \gamma) \left(\begin{bmatrix} it & \\ & z \end{bmatrix} \right) = 0$ for all $\gamma \in \mathrm{Sp}(4, \mathbb{Z})$ and $z \in \mathfrak{H}_1$.

For further background see, for example, [PY].

3. A TWISTING OPERATOR ON PARAMODULAR CUSP FORMS

3.1. Statement of the main theorem. Let Q be a 2×2 symmetric matrix, and let P be an invertible 2×2 matrix. Then the matrices

$$U(Q) = \begin{bmatrix} 1 & Q \\ & 1 \end{bmatrix} \quad \text{and} \quad A(P) = \begin{bmatrix} P & \\ & {}_t P^{-1} \end{bmatrix}.$$

are in $\mathrm{GSp}(4)$. In the following theorem, we extend the slash operator $|_k$ to formal \mathbb{C} -linear combinations of elements of $\mathrm{GSp}(4, \mathbb{R})^+$.

Theorem 3.1. *Let N and k be positive integers, p a prime with $p \nmid N$, and χ a nontrivial quadratic Dirichlet character mod p . Then, the local twisting map from Theorem 1.2 of [JR2] induces a linear map*

$$\mathcal{T}_\chi : S_k(\Gamma^{\mathrm{para}}(N)) \rightarrow S_k(\Gamma^{\mathrm{para}}(Np^4)).$$

If $F \in S_k(\Gamma^{\mathrm{para}}(N))$, then this map is given by the formula

$$\mathcal{T}_\chi(F) = \sum_{i=1}^{14} F|_k \mathcal{T}_\chi^i, \tag{5}$$

where

$$\mathcal{T}_\chi^1 = p^{-11} \sum_{\substack{a,b,x \in (\mathbb{Z}/p^3\mathbb{Z})^\times \\ z \in \mathbb{Z}/p^4\mathbb{Z}}} \chi(ab) U \left(\begin{bmatrix} zp^{-4} & -bp^{-2} \\ -bp^{-2} & -x^{-1}p^{-1} \end{bmatrix} \right) A \left(\begin{bmatrix} 1 & (a+xb)p^{-1} \\ & 1 \end{bmatrix} \right),$$

$$\mathcal{T}_\chi^2 = p^{-11} \sum_{\substack{a,b \in (\mathbb{Z}/p^3\mathbb{Z})^\times \\ x,y \in (\mathbb{Z}/p^3\mathbb{Z})^\times \\ x,y \neq 1(p)}} \chi(abxy) U \left(\begin{bmatrix} -ab(1 - (1-y)^{-1}x)p^{-3} & -ap^{-2} \\ -ap^{-2} & -ab^{-1}(1-x)^{-1}p^{-1} \end{bmatrix} \right) A \left(\begin{bmatrix} p & bp^{-1} \\ & 1 \end{bmatrix} \right),$$

$$\mathcal{T}_\chi^3 = p^{-6} \sum_{\substack{a \in (\mathbb{Z}/p^2\mathbb{Z})^\times \\ b \in (\mathbb{Z}/p^3\mathbb{Z})^\times \\ z \in (\mathbb{Z}/p\mathbb{Z})^\times \\ z \neq 1(p)}} \chi(b(1-z)) U \left(\begin{bmatrix} -bp^{-3} & ap^{-2} \\ ap^{-2} & -a^2b^{-1}zp^{-1} \end{bmatrix} \right) A \left(\begin{bmatrix} p & \\ & 1 \end{bmatrix} \right),$$

$$\mathcal{T}_\chi^4 = p^{-10} \sum_{\substack{a \in \mathbb{Z}/p^4\mathbb{Z} \\ b \in (\mathbb{Z}/p^3\mathbb{Z})^\times \\ x \in (\mathbb{Z}/p^4\mathbb{Z})^\times}} \chi(b) U \left(\begin{bmatrix} (ax-bp)p^{-4} & ap^{-2} \\ ap^{-2} & \end{bmatrix} \right) A \left(\begin{bmatrix} p & xp^{-2} \\ & 1 \end{bmatrix} \right),$$

$$\mathcal{T}_\chi^5 = p^{-9} \sum_{\substack{a,b \in (\mathbb{Z}/p^3\mathbb{Z})^\times \\ x \in \mathbb{Z}/p^3\mathbb{Z}}} \chi(b) U \left(\begin{bmatrix} (ax-b)p^{-3} & ap^{-2} \\ ap^{-2} & \end{bmatrix} \right) A \left(\begin{bmatrix} p & xp^{-1} \\ & 1 \end{bmatrix} \right),$$

$$\mathcal{T}_\chi^6 = p^{-6} \sum_{\substack{a,b \in (\mathbb{Z}/p^2\mathbb{Z})^\times \\ x \in (\mathbb{Z}/p\mathbb{Z})^\times}} \chi(bx) U \left(\begin{bmatrix} b(1+xp)p^{-2} & ap^{-2} \\ ap^{-2} & a^2b^{-1}p^{-2} \end{bmatrix} \right) A \left(\begin{bmatrix} p^2 & \\ & 1 \end{bmatrix} \right),$$

$$\mathcal{T}_\chi^7 = p^{-7} \sum_{\substack{a \in (\mathbb{Z}/p^3\mathbb{Z})^\times \\ b \in (\mathbb{Z}/p\mathbb{Z})^\times \\ z \in \mathbb{Z}/p^4\mathbb{Z}}} \chi(ab) U \left(\begin{bmatrix} zp^{-4} & bp^{-1} \\ bp^{-1} & \end{bmatrix} \right) A \left(\begin{bmatrix} 1 & -ap^{-1} \\ & p \end{bmatrix} \right),$$

$$\mathcal{T}_\chi^8 = p^{-9} \sum_{\substack{a,b,z \in (\mathbb{Z}/p^3\mathbb{Z})^\times \\ z \neq 1(p)}} \chi(abz(1-z)) U \left(\begin{bmatrix} ab(1-z)p^{-3} & ap^{-1} \\ ap^{-1} & \end{bmatrix} \right) A \left(\begin{bmatrix} p & bp^{-1} \\ & p \end{bmatrix} \right),$$

$$\mathcal{T}_\chi^9 = p^{-6} \sum_{\substack{a \in (\mathbb{Z}/p^2\mathbb{Z})^\times \\ b,x \in (\mathbb{Z}/p\mathbb{Z})^\times}} \chi(b) U \left(\begin{bmatrix} bp^{-1} & \\ & xp^{-1} \end{bmatrix} \right) A \left(\begin{bmatrix} p^2 & a \\ & p \end{bmatrix} \right),$$

$$\mathcal{T}_\chi^{10} = p^{-6} \sum_{\substack{a \in (\mathbb{Z}/p^2\mathbb{Z})^\times \\ b \in (\mathbb{Z}/p\mathbb{Z})^\times}} \chi(b) U \left(\begin{bmatrix} bp^{-1} & \\ & \end{bmatrix} \right) A \left(\begin{bmatrix} p^2 & a \\ & p^2 \end{bmatrix} \right),$$

$$\mathcal{T}_\chi^{11} = p^{-10} \sum_{\substack{a \in (\mathbb{Z}/p^2\mathbb{Z})^\times \\ b \in (\mathbb{Z}/p^4\mathbb{Z})^\times \\ x \in \mathbb{Z}/p^3\mathbb{Z} \\ z \in \mathbb{Z}/p^4\mathbb{Z}}} \chi(ab) U \left(\begin{bmatrix} zp^{-4} & (ap+xb)p^{-3} \\ (ap+xb)p^{-3} & xp^{-2} \end{bmatrix} \right) A \left(\begin{bmatrix} 1 & bp^{-2} \\ & p^{-1} \end{bmatrix} \right),$$

$$\begin{aligned}
\mathcal{T}_\chi^{12} &= p^{-12} \sum_{\substack{y \in \mathbb{Z}/p^4\mathbb{Z} \\ a \in (\mathbb{Z}/p^4\mathbb{Z})^\times \\ b, z \in (\mathbb{Z}/p^3\mathbb{Z})^\times \\ z \neq 1(p)}} \chi(abz(1-z)) U \left(\begin{bmatrix} a(y-b(1-z)p)p^{-4} & yp^{-3} \\ yp^{-3} & a^{-1}(y+bp)p^{-2} \end{bmatrix} \right) A \left(\begin{bmatrix} p & ap^{-2} \\ & p^{-1} \end{bmatrix} \right), \\
\mathcal{T}_\chi^{13} &= p^{-6} \sum_{\substack{a \in (\mathbb{Z}/p^2\mathbb{Z})^\times \\ b \in (\mathbb{Z}/p^3\mathbb{Z})^\times \\ x \in (\mathbb{Z}/p\mathbb{Z})^\times}} \chi(bx) U \left(\begin{bmatrix} b(1+x)p^{-1} & ap^{-2} \\ ap^{-2} & a^2b^{-1}p^{-3} \end{bmatrix} \right) A \left(\begin{bmatrix} p^2 & \\ & p^{-1} \end{bmatrix} \right), \\
\mathcal{T}_\chi^{14} &= p^{-6} \sum_{\substack{a \in (\mathbb{Z}/p^2\mathbb{Z})^\times \\ b \in (\mathbb{Z}/p\mathbb{Z})^\times \\ x \in \mathbb{Z}/p^4\mathbb{Z}}} \chi(b) U \left(\begin{bmatrix} bp^{-1} & ap^{-2} \\ ap^{-2} & xp^{-4} \end{bmatrix} \right) A \left(\begin{bmatrix} p^2 & \\ & p^{-2} \end{bmatrix} \right).
\end{aligned}$$

Moreover, for every prime $\ell \neq p$ we have the following commutation relations for the Hecke operators (9) and Atkin-Lehner operator (12):

$$\begin{aligned}
T(1, 1, \ell, \ell) \mathcal{T}_\chi &= \chi(\ell) \mathcal{T}_\chi T(1, 1, \ell, \ell), \\
T(1, \ell, \ell, \ell^2) \mathcal{T}_\chi &= \mathcal{T}_\chi T(1, \ell, \ell, \ell^2), \\
\mathcal{T}_\chi(F)|_k U_\ell &= \chi(\ell)^{\text{val}_\ell(N)} \mathcal{T}_\chi(F|_k U_\ell),
\end{aligned}$$

for $F \in S_k(\Gamma^{\text{para}}(N))$.

3.2. Twisted automorphic forms. In order to present the proof of Theorem 3.1 we must explain the connection between Siegel modular forms and automorphic forms on $\text{GSp}(4, \mathbb{A})$. Let k be a positive integer, and let $\mathcal{F} = \{K_\ell\}_\ell$, where ℓ runs over the finite primes of \mathbb{Q} , be a family of compact, open subgroups of $\text{GSp}(4, \mathbb{Q}_\ell)$ such that $K_\ell = \text{GSp}(4, \mathbb{Z}_\ell)$ for almost all ℓ and $\lambda(K_\ell) = \mathbb{Z}_\ell^\times$ for all ℓ . To \mathcal{F} and k we will associate a space of automorphic forms on $\text{GSp}(4, \mathbb{A})$ and a space of Siegel modular forms of degree two. Set

$$K_{\mathcal{F}} = \prod_{\ell < \infty} K_\ell \subset \text{GSp}(4, \mathbb{A}_f).$$

Since $\lambda(K_\ell) = \mathbb{Z}_\ell^\times$ for all finite ℓ , strong approximation for $\text{Sp}(4)$ implies that

$$\text{GSp}(4, \mathbb{A}) = \text{GSp}(4, \mathbb{Q}) \text{GSp}(4, \mathbb{R})^+ K_{\mathcal{F}}.$$

Let $\chi : \mathbb{A}^\times \rightarrow \mathbb{C}^\times$ be a quadratic Hecke character, and let $\chi = \prod_{\ell \leq \infty} \chi_\ell$ be the decomposition of χ as a product of local characters. Then $\chi_\infty(\mathbb{R}_{>0}^\times) = 1$. Also let

$$K_\infty = \left\{ \begin{bmatrix} A & B \\ -B & A \end{bmatrix} \in \text{GL}(4, \mathbb{R}) : {}^tAA + {}^tBB = \mathbf{1}, {}^tAB = {}^tBA \right\}.$$

We define $S_k(K_{\mathcal{F}}, \chi)$ to be the space of continuous functions $\Phi : \text{GSp}(4, \mathbb{A}) \rightarrow \mathbb{C}$ such that

- (1) $\Phi(\rho g) = \Phi(g)$ for all $\rho \in \text{GSp}(4, \mathbb{Q})$ and $g \in \text{GSp}(4, \mathbb{A})$;
- (2) $\Phi(gz) = \Phi(g)$ for all $z \in \mathbb{A}^\times$ and $g \in \text{GSp}(4, \mathbb{A})$;
- (3) $\Phi(g\kappa_\ell) = \chi_\ell(\lambda(\kappa_\ell))\Phi(g)$ for all $\kappa_\ell \in K_\ell$, $g \in \text{GSp}(4, \mathbb{A})$ and finite primes ℓ of \mathbb{Q} ;
- (4) $\Phi(gk_\infty) = j(k_\infty, I)^{-k}\Phi(g)$ for all $k_\infty \in K_\infty$ and $g \in \text{GSp}(4, \mathbb{A})$;
- (5) For any proper parabolic subgroup P of $\text{GSp}(4)$

$$\int_{N_P(\mathbb{Q}) \backslash N_P(\mathbb{A})} \Phi(n g) dn = 0$$

- for all $g \in \mathrm{GSp}(4, \mathbb{A})$; here N_P is the unipotent radical of P ;
- (6) For any $g_f \in \mathrm{GSp}(4, \mathbb{A}_f)$, the function $\mathrm{GSp}(4, \mathbb{R})^+ \rightarrow \mathbb{C}$ defined by $g_\infty \mapsto \Phi(g_f g_\infty)$ is smooth and is annihilated by $\mathfrak{p}_{\mathbb{C}}^-$, where we refer to Section 3.5 of [AS] for the definition of $\mathfrak{p}_{\mathbb{C}}^-$.

On the other hand, to \mathcal{F} we associate a subgroup of $\mathrm{Sp}(4, \mathbb{Q})$ that is commensurable with $\mathrm{GSp}(4, \mathbb{Z})$,

$$\Gamma_{\mathcal{F}} = \mathrm{GSp}(4, \mathbb{Q}) \cap \mathrm{GSp}(4, \mathbb{R})^+ \prod_{\ell < \infty} K_{\ell}.$$

We define $S_k(\Gamma_{\mathcal{F}})$ to be the complex vector space of functions $F : \mathfrak{H}_2 \rightarrow \mathbb{C}$ such that

- (1) F is holomorphic;
- (2) $F|_k \gamma = F$ for all $\gamma \in \Gamma_{\mathcal{F}}$;
- (3) $\lim_{t \rightarrow \infty} (F|_k \gamma) \left(\begin{bmatrix} it & \\ & z \end{bmatrix} \right) = 0$ for all $\gamma \in \mathrm{Sp}(4, \mathbb{Z})$ and $z \in \mathfrak{H}_1$.

Lemma 3.2. *Let χ, \mathcal{F} and k be as above. For $F \in S_k(\Gamma_{\mathcal{F}})$, define $\Phi_F : \mathrm{GSp}(4, \mathbb{A}) \rightarrow \mathbb{C}$ by*

$$\Phi_F(\rho h \kappa) = (F|_k h)(I) = \lambda(h)^k j(h, I)^{-k} \cdot \prod_{\ell < \infty} \chi_{\ell}(\lambda(\kappa_{\ell})) \cdot F(h\langle I \rangle)$$

for $\rho \in \mathrm{GSp}(4, \mathbb{Q})$, $h \in \mathrm{GSp}(4, \mathbb{R})^+$, $\kappa \in K_{\mathcal{F}} = \prod_{\ell < \infty} K_{\ell}$. Then Φ_F is a well-defined element of $S_k(K_{\mathcal{F}}, \chi)$, so that there is a complex linear map

$$S_k(\Gamma_{\mathcal{F}}) \longrightarrow S_k(K_{\mathcal{F}}, \chi). \quad (6)$$

Conversely, let $\Phi \in S_k(K_{\mathcal{F}}, \chi)$. Define $F_{\Phi} : \mathfrak{H}_2 \rightarrow \mathbb{C}$ by

$$F_{\Phi}(Z) = \lambda(h)^{-k} j(h, I)^k \Phi(h_{\infty})$$

for $Z \in \mathfrak{H}_2$ with $h \in \mathrm{GSp}(4, \mathbb{R})^+$ with $h\langle I \rangle = Z$. Then F_{Φ} is well-defined and contained in $S_k(\Gamma_{\mathcal{F}})$, so that there is complex linear map

$$S_k(K_{\mathcal{F}}, \chi) \longrightarrow S_k(\Gamma_{\mathcal{F}}). \quad (7)$$

Moreover, the maps (6) and (7) are inverses of each other.

Proof. To prove that Φ_F is well defined, suppose that $\rho h \kappa = \rho' h' \kappa'$ for $\rho, \rho' \in \mathrm{GSp}(4, \mathbb{Q})$, $h, h' \in \mathrm{GSp}(4, \mathbb{R})^+$, and $\kappa, \kappa' \in \prod_{\ell < \infty} K_{\ell}$. Comparing components, we have that $\rho h = \rho' h' \in \mathrm{GSp}(4, \mathbb{R})^+$ and $\rho \kappa_{\ell} = \rho' \kappa'_{\ell} \in \mathrm{GSp}(4, \mathbb{Q}_{\ell})$ for $\ell < \infty$. Therefore $\rho_0 \rho'^{-1} \rho \in \Gamma_{\mathcal{F}}$ and we have that

$$\begin{aligned} & \lambda(h')^k j(h', I)^{-k} \prod_{\ell < \infty} \chi_{\ell}(\lambda(\kappa'_{\ell})) F(h'\langle I \rangle) \\ &= \lambda(\rho_0 h)^k j(\rho_0 h, I)^{-k} \prod_{\ell < \infty} \chi_{\ell}(\lambda(\rho_0 \kappa_{\ell})) F((\rho_0 h)\langle I \rangle) \\ &= \lambda(\rho_0)^k \lambda(h)^k j(\rho_0, h\langle I \rangle)^{-k} j(h, I)^{-k} \prod_{\ell < \infty} \chi_{\ell}(\lambda(\kappa_{\ell})) F(\rho_0 \langle h\langle I \rangle \rangle) \\ &= \lambda(h)^k j(h, I)^{-k} \prod_{\ell < \infty} \chi_{\ell}(\lambda(\kappa_{\ell})) (F|_k \rho_0)(h\langle I \rangle) \\ &= \lambda(h)^k j(h, I)^{-k} \prod_{\ell < \infty} \chi_{\ell}(\lambda(\kappa_{\ell})) F(h\langle I \rangle). \end{aligned}$$

This shows that Φ_F is well-defined. Straightforward calculations show that the function also satisfies first four conditions in the definition of $S_k(K_{\mathcal{F}}, \chi)$. The proofs of the fifth and sixth conditions are

similar to the proofs of Lemma 5 and Lemma 7, respectively, in [AS]. Similar calculations show that $F_\Phi \in S_k(\Gamma_{\mathcal{F}})$, for Φ in $S_k(K_{\mathcal{F}}, \chi)$. Finally, it is clear that these two maps are inverses of each other. \square

Lemma 3.3. *Let k be a positive integer and let $\mathcal{F}_1 = \{K_\ell^1\}_\ell$ and $\mathcal{F}_2 = \{K_\ell^2\}_\ell$, where ℓ runs over the finite primes of \mathbb{Q} , be families of compact, open subgroups of $\mathrm{GSp}(4, \mathbb{Q}_\ell)$ such that $K_\ell^1 = K_\ell^2 = \mathrm{GSp}(4, \mathbb{Z}_\ell)$ for almost all ℓ and $\lambda(K_\ell^1) = \lambda(K_\ell^2) = \mathbb{Z}_\ell^\times$ for all ℓ . Let χ_1 and χ_2 be quadratic Hecke characters. Suppose that there is a linear map*

$$T : S_k(K_{\mathcal{F}_1}, \chi_1) \rightarrow S_k(K_{\mathcal{F}_2}, \chi_2)$$

given by a right translation formula at the p th place,

$$T(\Phi_1) = \sum_{i=1}^t c_i R(B_{i,p}) \Phi_1,$$

for $\Phi_1 \in S_k(K_{\mathcal{F}_1}, \chi_1)$. Here $c_i \in \mathbb{C}^\times$ and $B_i \in \mathrm{Sp}(4, \mathbb{Q})^+$ are such that $B_i \in K_\ell^1$ for $\ell \neq p$, $i \in \{1, \dots, t\}$. Then the composition, \mathcal{T} ,

$$\begin{array}{ccc} S_k(\Gamma_{\mathcal{F}_1}) & \xrightarrow{\mathcal{T}} & S_k(\Gamma_{\mathcal{F}_2}) \\ \wr \downarrow & & \uparrow \wr \\ S_k(K_{\mathcal{F}_1}, \chi_1) & \xrightarrow{T} & S_k(K_{\mathcal{F}_2}, \chi_2) \end{array} \quad (8)$$

is given by the formula

$$\mathcal{T}(F) = \sum_{i=1}^t c_i \cdot F|_k(B_i)^{-1}$$

for $F \in S_k(\Gamma_{\mathcal{F}_1})$.

Proof. Let $F \in S_k(\Gamma_{\mathcal{F}_1})$. By the isomorphism (6) for the family \mathcal{F}_1 with character χ_1 we have $\Phi_1 = \Phi_F \in S_k(K_{\mathcal{F}_1}, \chi_1)$. Using the isomorphism (7) for the family \mathcal{F}_2 with character χ_2 , we calculate the composition $F_{T(\Phi_1)}$. Let $Z \in \mathfrak{H}_2$ and let $h \in \mathrm{GSp}(4, \mathbb{R})^+$ be such that $h\langle I \rangle = Z$. Then, using that $\lambda(B_i) = 1$, we have that

$$\begin{aligned} F_{T(\Phi_1)}(Z) &= \lambda(h)^{-k} j(h, I)^k \sum_{i=1}^t c_i \Phi_1(h_\infty B_{i,p}) \\ &= \lambda(h)^{-k} j(h, I)^k \sum_{i=1}^t c_i \Phi_1(B_i^{-1} h_\infty B_{i,p}) \\ &= \lambda(h)^{-k} j(h, I)^k \sum_{i=1}^t c_i \Phi_1(B_{i,\infty}^{-1} h_\infty) \\ &= \lambda(h)^{-k} j(h, I)^k \sum_{i=1}^t c_i \lambda(B_i^{-1} h)^k j(B_i^{-1} h, I)^{-k} F((B_i^{-1} h)\langle I \rangle) \\ &= j(h, I)^k \sum_{i=1}^t c_i j(B_i^{-1}, Z)^{-k} j(h, I)^{-k} F(B_i^{-1} \langle Z \rangle) \\ &= \sum_{i=1}^t c_i j(B_i^{-1}, Z)^{-k} F(B_i^{-1} \langle Z \rangle) \end{aligned}$$

$$= \sum_{i=1}^t c_i (F|_k B_i^{-1})(Z).$$

Hence,

$$\mathcal{T}(F) = F_{T(\Phi_1)} = \sum_{i=1}^t c_i \cdot F|_k B_i^{-1}.$$

This completes the proof. \square

3.3. Paramodular Hecke operators. Throughout this section, let M and k be positive integers, χ a quadratic Hecke character, and ℓ a fixed rational prime. We turn now to the paramodular family $\mathcal{F} = \{K^{\text{para}}(\ell^{\text{val}_\ell(M)})\}_\ell$. We will determine the explicit relationship between the Hecke operators for Siegel paramodular forms and those for twisted automorphic forms. Let ℓ be a rational prime and define the Hecke operators $T(1, 1, \ell, \ell)$ and $T(1, \ell, \ell, \ell^2)$ on $S_k(\Gamma^{\text{para}}(M))$ by

$$T(1, 1, \ell, \ell)F = p^{k-3} \sum_i F|_k a_i, \quad T(1, \ell, \ell, \ell^2)F = p^{2(k-3)} \sum_j F|_k b_j \quad (9)$$

where

$$\Gamma^{\text{para}}(M) \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \ell & \\ & & & \ell \end{bmatrix} \Gamma^{\text{para}}(M) = \bigsqcup \Gamma^{\text{para}}(M) a_i \quad (10)$$

and

$$\Gamma^{\text{para}}(M) \begin{bmatrix} 1 & & & \\ & \ell & & \\ & & \ell^2 & \\ & & & \ell \end{bmatrix} \Gamma^{\text{para}}(M) = \bigsqcup \Gamma^{\text{para}}(M) b_j, \quad (11)$$

are disjoint decompositions. We define the Atkin-Lehner involution U_ℓ on $S_k(\Gamma^{\text{para}}(M))$ as follows. Choose a matrix $\gamma_\ell \in \text{Sp}(4, \mathbb{Z})$ such that

$$\gamma_\ell \equiv \begin{bmatrix} & & & 1 \\ & & 1 & \\ & -1 & & \\ -1 & & & \end{bmatrix} \pmod{\ell^{\text{val}_\ell(M)}} \quad \text{and} \quad \gamma_\ell \equiv \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \pmod{M\ell^{-\text{val}_\ell(M)}}.$$

Then,

$$U_\ell = \gamma_\ell \begin{bmatrix} \ell^{\text{val}_\ell(M)} & & & \\ & \ell^{\text{val}_\ell(M)} & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \quad (12)$$

normalizes $\Gamma^{\text{para}}(M)$ and U_ℓ^2 is contained in $\ell^{\text{val}_\ell(M)} \Gamma^{\text{para}}(M)$, implying that $F \mapsto F|_k U_\ell$ is indeed an involution of $S_K(\Gamma^{\text{para}}(M))$. The proof of the following lemma provides explicit forms for the representatives a_i and b_j from (10) and (11).

Lemma 3.4. *Let M be a positive integer, let ℓ be a prime and set $r = \text{val}_\ell(M)$. There exist finite disjoint decompositions*

$$K^{\text{para}}(\ell^r) \begin{bmatrix} \ell & & & \\ & \ell & & \\ & & 1 & \\ & & & 1 \end{bmatrix} K^{\text{para}}(\ell^r) = \bigsqcup g_i K^{\text{para}}(\ell^r)$$

and

$$K^{\text{para}}(\ell^r) \begin{bmatrix} \ell^2 & & & \\ & \ell & & \\ & & 1 & \\ & & & \ell \end{bmatrix} K^{\text{para}}(\ell^r) = \bigsqcup h_j K^{\text{para}}(\ell^r)$$

such that

$$\Gamma^{\text{para}}(M) \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \ell & \\ & & & \ell \end{bmatrix} \Gamma^{\text{para}}(M) = \bigsqcup \Gamma^{\text{para}}(M) \ell g_i^{-1}$$

and

$$\Gamma^{\text{para}}(M) \begin{bmatrix} 1 & & & \\ & \ell & & \\ & & \ell^2 & \\ & & & \ell \end{bmatrix} \Gamma^{\text{para}}(M) = \bigsqcup \Gamma^{\text{para}}(M) \ell^2 h_j^{-1}.$$

The explicit forms of the representatives are given in the proof.

Proof. We first consider the case when $r = 0$ so that $K^{\text{para}}(\ell^r) = \text{GSp}(4, \mathbb{Z}_\ell)$. From Section 6.1 of [RS] we have the decompositions

$$\begin{aligned} \text{GSp}(4, \mathbb{Z}_\ell) \begin{bmatrix} \ell & & & \\ & \ell & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \text{GSp}(4, \mathbb{Z}_\ell) &= \bigsqcup_{x, y, z \in \mathbb{Z}/\ell\mathbb{Z}} \begin{bmatrix} 1 & & z & y \\ & 1 & y & x \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} \ell & & & \\ & \ell & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \text{GSp}(4, \mathbb{Z}_\ell) \\ &\sqcup \bigsqcup_{x, z \in \mathbb{Z}/\ell\mathbb{Z}} \begin{bmatrix} 1 & x & & \\ & 1 & & \\ & & 1 & \\ & & & -x & 1 \end{bmatrix} \begin{bmatrix} 1 & & z & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} \ell & & & \\ & 1 & & \\ & & 1 & \\ & & & \ell \end{bmatrix} \text{GSp}(4, \mathbb{Z}_\ell) \\ &\sqcup \bigsqcup_{x \in \mathbb{Z}/\ell\mathbb{Z}} \begin{bmatrix} 1 & & & \\ & 1 & & x \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & \ell & & \\ & & \ell & \\ & & & 1 \end{bmatrix} \text{GSp}(4, \mathbb{Z}_\ell) \\ &\sqcup \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \ell & \\ & & & \ell \end{bmatrix} \text{GSp}(4, \mathbb{Z}_\ell), \end{aligned}$$

and

$$\begin{aligned}
\mathrm{GSp}(4, \mathbb{Z}_\ell) \begin{bmatrix} \ell^2 & & & \\ & \ell & & \\ & & 1 & \\ & & & \ell \end{bmatrix} \mathrm{GSp}(4, \mathbb{Z}_\ell) = \\
\bigcup_{\substack{z \in \mathbb{Z}/\ell^2\mathbb{Z} \\ x, y \in \mathbb{Z}/\ell\mathbb{Z}}} \begin{bmatrix} 1 & x & & \\ & 1 & & \\ & & 1 & \\ & & -x & 1 \end{bmatrix} \begin{bmatrix} 1 & & z & y \\ & 1 & y & 1 \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} \ell^2 & & & \\ & \ell & & \\ & & 1 & \\ & & & \ell \end{bmatrix} \mathrm{GSp}(4, \mathbb{Z}_\ell) \\
\sqcup \bigcup_{\substack{c \in \mathbb{Z}/\ell\mathbb{Z} \\ d \in \mathbb{Z}/\ell^2\mathbb{Z}}} \begin{bmatrix} 1 & & c & d \\ & 1 & 1 & \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} \ell & & & \\ & \ell^2 & & \\ & & \ell & \\ & & & 1 \end{bmatrix} \mathrm{GSp}(4, \mathbb{Z}_\ell) \\
\sqcup \bigcup_{x \in \mathbb{Z}/\ell\mathbb{Z}} \begin{bmatrix} 1 & x & & \\ & 1 & & \\ & & 1 & \\ & & -x & 1 \end{bmatrix} \begin{bmatrix} \ell & & & \\ & 1 & & \\ & & \ell & \\ & & & \ell^2 \end{bmatrix} \mathrm{GSp}(4, \mathbb{Z}_\ell) \\
\sqcup \begin{bmatrix} 1 & & & \\ & \ell & & \\ & & \ell^2 & \\ & & & \ell \end{bmatrix} \mathrm{GSp}(4, \mathbb{Z}_\ell) \\
\sqcup \bigcup_{d \in (\mathbb{Z}/\ell\mathbb{Z})^\times} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & d\ell & 1 \end{bmatrix} \begin{bmatrix} \ell & & & \\ & 1 & & \\ & & \ell & \\ & & & \ell^2 \end{bmatrix} \mathrm{GSp}(4, \mathbb{Z}_\ell) \\
\sqcup \bigcup_{u \in (\mathbb{Z}/\ell\mathbb{Z})^\times} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & u\ell & & 1 \end{bmatrix} \begin{bmatrix} \ell & & & \\ & 1 & & \\ & & \ell & \\ & & & \ell^2 \end{bmatrix} \mathrm{GSp}(4, \mathbb{Z}_\ell) \\
\sqcup \bigcup_{u, \lambda \in (\mathbb{Z}/\ell\mathbb{Z})^\times} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & u\ell & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ \lambda & 1 & & \\ & & 1 & -\lambda \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & \ell & & \\ & & \ell^2 & \\ & & & \ell \end{bmatrix} \mathrm{GSp}(4, \mathbb{Z}_\ell).
\end{aligned}$$

For $r \geq 1$, we again refer to [RS] Section 6.1 to deduce the decompositions

$$\begin{aligned}
K^{\mathrm{para}}(\ell^r) \begin{bmatrix} \ell & & & \\ & \ell & & \\ & & 1 & \\ & & & 1 \end{bmatrix} K^{\mathrm{para}}(\ell^r) = \bigcup_{x, y, z \in \mathbb{Z}/\ell\mathbb{Z}} \begin{bmatrix} 1 & & z\ell^{-r} & y \\ & 1 & y & x \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} \ell & & & \\ & \ell & & \\ & & 1 & \\ & & & 1 \end{bmatrix} K^{\mathrm{para}}(\ell^r) \\
\sqcup \bigcup_{x, z \in \mathbb{Z}/\ell\mathbb{Z}} \begin{bmatrix} 1 & x & z\ell^{-r} & \\ & 1 & & \\ & & 1 & \\ & & -x & 1 \end{bmatrix} \begin{bmatrix} \ell & & & \\ & 1 & & \\ & & 1 & \\ & & & \ell \end{bmatrix} K^{\mathrm{para}}(\ell^r)
\end{aligned}$$

$$\begin{aligned} & \bigsqcup_{x,y \in \mathbb{Z}/\ell\mathbb{Z}} \begin{bmatrix} & & -\ell^{-r} \\ & 1 & \\ \ell^r & & 1 \end{bmatrix} \begin{bmatrix} 1 & & y \\ & 1 & x \\ & & 1 \end{bmatrix} \begin{bmatrix} \ell & & \\ & \ell & \\ & & 1 \end{bmatrix} K^{\text{para}}(\ell^r) \\ & \bigsqcup_{x \in \mathbb{Z}/\ell\mathbb{Z}} \begin{bmatrix} & & -\ell^{-r} \\ & 1 & \\ \ell^r & & 1 \end{bmatrix} \begin{bmatrix} 1 & x & \\ & 1 & \\ & & 1 \end{bmatrix} \begin{bmatrix} \ell & & \\ & 1 & \\ & & 1 \end{bmatrix} K^{\text{para}}(\ell^r) \end{aligned}$$

and

$$\begin{aligned} & K^{\text{para}}(\ell^r) \begin{bmatrix} \ell^2 & & \\ & \ell & \\ & & 1 \end{bmatrix} K^{\text{para}}(\ell^r) = \\ & \bigsqcup_{\substack{x,y \in \mathbb{Z}/\ell\mathbb{Z} \\ z \in \mathbb{Z}/\ell^2\mathbb{Z}}} \begin{bmatrix} 1 & x & \\ & 1 & \\ & & 1 \end{bmatrix} \begin{bmatrix} 1 & & z\ell^{-r} & y \\ & 1 & y & \\ & & 1 & 1 \end{bmatrix} \begin{bmatrix} \ell^2 & & \\ & \ell & \\ & & 1 \end{bmatrix} K^{\text{para}}(\ell^r) \\ & \bigsqcup_{x,y,z \in \mathbb{Z}/\ell\mathbb{Z}} \begin{bmatrix} & & -\ell^{-r} \\ & 1 & \\ \ell^r & & 1 \end{bmatrix} \begin{bmatrix} 1 & x & \\ & 1 & \\ & & 1 \end{bmatrix} \begin{bmatrix} 1 & & z\ell^{-r+1} & y \\ & 1 & y & \\ & & 1 & 1 \end{bmatrix} \begin{bmatrix} \ell^2 & & \\ & \ell & \\ & & 1 \end{bmatrix} K^{\text{para}}(\ell^r). \end{aligned}$$

The explicit form of these decompositions implies that we can choose representatives g_i and h_j such that

$$\begin{aligned} & K^{\text{para}}(\ell^r) \begin{bmatrix} \ell & & \\ & \ell & \\ & & 1 \end{bmatrix} K^{\text{para}}(\ell^r) = \bigsqcup_{i=1}^D g_i K^{\text{para}}(\ell^r), \\ & K^{\text{para}}(\ell^r) \begin{bmatrix} \ell^2 & & \\ & \ell & \\ & & 1 \end{bmatrix} K^{\text{para}}(\ell^r) = \bigsqcup_{j=1}^{D'} h_j K^{\text{para}}(\ell^r), \end{aligned}$$

and

$$\ell g_i^{-1} \in \Gamma^{\text{para}}(M) \begin{bmatrix} 1 & & \\ & 1 & \\ & & \ell \end{bmatrix} \Gamma^{\text{para}}(M), \quad \ell^2 h_j^{-1} \in \Gamma^{\text{para}}(M) \begin{bmatrix} 1 & & \\ & \ell & \\ & & \ell^2 \end{bmatrix} \Gamma^{\text{para}}(M),$$

for $i \in \{1, \dots, D\}$ and $j \in \{1, \dots, D'\}$. For this, it is useful to note that $\Gamma^{\text{para}}(M)$ contains several symmetry elements:

$$\begin{bmatrix} 1 & & & \\ & & & 1 \\ & & 1 & \\ & -1 & & \end{bmatrix}, \quad \begin{bmatrix} & & -M^{-1} \\ & 1 & \\ M & & \\ & & 1 \end{bmatrix},$$

and in the case that $r = 0$, the element $\begin{bmatrix} A & \\ & {}_t A^{-1} \end{bmatrix}$ where $A \in \Gamma_0(M) \subset \mathrm{SL}(2, \mathbb{Z})$ and $A \begin{bmatrix} 1 & \\ & \ell \end{bmatrix} A^{-1} = \begin{bmatrix} \ell & \\ & 1 \end{bmatrix}$. It follows that the cosets $\Gamma^{\mathrm{para}}(M)\ell g_i^{-1}$ are mutually disjoint and contained in the first double coset, and the cosets $\Gamma^{\mathrm{para}}(M)\ell^2 h_j^{-1}$ are mutually disjoint and contained in the second double coset. It remains to prove that the number of disjoint cosets in the first and second double cosets are D and D' , respectively. Suppose that

$$\Gamma^{\mathrm{para}}(M) \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \ell & \\ & & & \ell \end{bmatrix} \Gamma^{\mathrm{para}}(M) = \bigsqcup_{i=1}^d \Gamma^{\mathrm{para}}(M) g'_i$$

is a disjoint decomposition. We have already shown that $d \geq D$; we need to prove that $D \geq d$. We have

$$K^{\mathrm{para}}(\ell^r) \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \ell & \\ & & & \ell \end{bmatrix} K^{\mathrm{para}}(\ell^r) \supset K^{\mathrm{para}}(\ell^r) \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \ell & \\ & & & \ell \end{bmatrix} \Gamma^{\mathrm{para}}(M) = \bigcup_{i=1}^d K^{\mathrm{para}}(\ell^r) g'_i.$$

We claim that these cosets are disjoint. For suppose $K^{\mathrm{para}}(\ell^r) g'_i = K^{\mathrm{para}}(\ell^r) g'_j$. This implies that $g'_i g'_j{}^{-1} \in K^{\mathrm{para}}(\ell^r)$. Since for any prime q with $q \neq \ell$ we have $g'_i g'_j{}^{-1} \in K^{\mathrm{para}}(q^{\mathrm{val}_q(M)})$ as all the elements of

$$\Gamma^{\mathrm{para}}(M) \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \ell & \\ & & & \ell \end{bmatrix} \Gamma^{\mathrm{para}}(M)$$

are contained in $K^{\mathrm{para}}(q^{\mathrm{val}_q(M)})$, we have $g'_i g'_j{}^{-1} \in \Gamma^{\mathrm{para}}(M)$. This implies that $i = j$. It follows that $D \geq d$. The proof that $d' = D'$ is similar. \square

Let V be the \mathbb{C} vector space of functions $\Phi : \mathrm{GSp}(4, \mathbb{A}) \rightarrow \mathbb{C}$ such that $\Phi \in V$ if and only if there exists a compact, open subgroup Γ_1 of $\mathrm{GSp}(4, \mathbb{Q}_\ell)$ such that $\Phi(gk) = \Phi(g)$ for $g \in \mathrm{GSp}(4, \mathbb{A})$ and $k \in \Gamma_1$, and $\Phi(gz) = \Phi(g)$ for $g \in \mathrm{GSp}(4, \mathbb{A})$ and $z \in \mathbb{A}^\times$. The group $\mathrm{GSp}(4, \mathbb{Q}_\ell)$ acts smoothly on V by right translation, and for this action, denoted by π , the center of $\mathrm{GSp}(4, \mathbb{Q}_\ell)$ acts trivially. Assume that χ_ℓ is unramified. Then $S_k(K_{\mathcal{F}}, \chi) \subset V(\mathrm{val}_\ell(M))$, where the last space consists of the vectors in V that are fixed by $K^{\mathrm{para}}(\ell^{\mathrm{val}_\ell(M)})$ as in [RS]. Let $T_{1,0}$ and $T_{0,1}$ be the local Hecke operators and $u_{\mathrm{val}_\ell(M)}$ be the local Atkin-Lehner operator acting on $V(\mathrm{val}_\ell(M))$ as in [RS]. These operators preserve the subspace $S_k(K_{\mathcal{F}}, \chi) \subset V(\mathrm{val}_\ell(M))$.

Lemma 3.5. *Assume that χ_ℓ is unramified. Let $\Phi \in S_k(K_{\mathcal{F}}, \chi)$ and let $F_\Phi \in S_k(\Gamma^{\mathrm{para}}(M))$ be the Siegel modular form corresponding to Φ under the isomorphism (7) of Lemma 3.2. Then,*

- (1) $T(1, 1, \ell, \ell) F_\Phi = \ell^{k-3} \chi_\ell(\ell) F_{T_{1,0}\Phi},$
- (2) $T(1, \ell, \ell, \ell^2) F_\Phi = \ell^{2(k-3)} F_{T_{0,1}\Phi},$
- (3) $F|_k U_\ell = \chi_\ell(\ell)^{\mathrm{val}_\ell(M)} F_{u_{\mathrm{val}_\ell(M)}\Phi}.$

Proof. Let $Z \in \mathfrak{H}_2$ and let $g_\infty \in \mathrm{Sp}(4, \mathbb{R})$ such that $g_\infty \langle I \rangle = Z$. For the first assertion, we will use the coset representatives from Lemma 3.4. Note that from the explicit forms given in the proof, we have that $g_i \in K^{\mathrm{para}}(q^{\mathrm{val}_q(M)})$ for primes $q \neq \ell$ and $\lambda(g_i) = \ell$ for each i . We calculate

$$(T(1, 1, \ell, \ell) F_\Phi)(Z) = \ell^{k-3} \sum_i (F_\Phi|_k \ell g_i^{-1})(Z)$$

$$\begin{aligned}
&= \ell^{k-3} \sum_i \lambda(\ell g_i^{-1})^k j(\ell g_i^{-1}, g_\infty \langle I \rangle)^{-k} F_\Phi(\ell g_i^{-1} g_\infty \langle I \rangle) \\
&= \ell^{k-3} \sum_i \lambda(\ell g_i^{-1})^k j(\ell g_i^{-1}, g_\infty \langle I \rangle)^{-k} \lambda(\ell g_i^{-1} g_\infty)^{-k} j(\ell g_i^{-1} g_\infty, I)^k \Phi((\ell g_i^{-1})_\infty g_\infty) \\
&= \ell^{k-3} \sum_i \lambda(g_\infty)^{-k} j(g_\infty, I)^k \Phi(g_{i,\infty}^{-1} g_\infty) \\
&= \ell^{k-3} \sum_i \lambda(g_\infty)^{-k} j(g_\infty, I)^k \Phi(g_{i,f} g_\infty) \\
&= \ell^{k-3} \sum_i \lambda(g_\infty)^{-k} j(g_\infty, I)^k \prod_{q \neq \ell} \chi_q(\lambda(g_i)) \Phi(g_\infty g_{i,\ell}) \\
&= \ell^{k-3} \chi_\ell(\ell) \sum_i \lambda(g_\infty)^{-k} j(g_\infty, I)^k \Phi(g_\infty g_{i,\ell}) \\
&= \ell^{k-3} \chi_\ell(\ell) \lambda(g_\infty)^{-k} j(g_\infty, I)^k (T_{1,0} \Phi)(g_\infty) \\
&= \ell^{k-3} \chi_\ell(\ell) (F_{T_{1,0}} \Phi)(Z).
\end{aligned}$$

The proof of the second and third assertions are similar. \square

3.4. Proof of the main theorem.

Proof of Theorem 3.1. We will use Lemma 3.3, Lemma 3.5, and Corollary 5.1. Let χ_1 be the trivial Hecke character, and let χ_2 be the Hecke character corresponding to χ , as in (4). Let $\mathcal{F}_1 = \{K^{\text{para}}(\ell^{\text{val}_\ell(N)})\}_\ell$ and $\mathcal{F}_2 = \{K^{\text{para}}(\ell^{\text{val}_\ell(Np^4)})\}_\ell$. Let V be the \mathbb{C} vector space of functions $\Phi : \text{GSp}(4, \mathbb{A}) \rightarrow \mathbb{C}$ such that $\Phi \in V$ if and only if there exists a compact, open subgroup Γ_1 of $\text{GSp}(4, \mathbb{Q}_p)$ such that $\Phi(gk) = \Phi(g)$ for $g \in \text{GSp}(4, \mathbb{A})$ and $k \in \Gamma_1$, and $\Phi(gz) = \Phi(g)$ for $g \in \text{GSp}(4, \mathbb{A})$ and $z \in \mathbb{A}^\times$. The group $\text{GSp}(4, \mathbb{Q}_p)$ acts smoothly on V by right translation, and for this action π the center of $\text{GSp}(4, \mathbb{Q}_p)$ acts trivially. Next, the operator $T_{\chi_p} : V(0) \rightarrow V(4, \chi_p)$ from Corollary 5.1 has the form

$$T_{\chi_p}(\Phi) = \sum_{i=1}^t c_i \cdot \pi(B_i) \Phi$$

where $c_i \in \mathbb{C}$ and $B_i \in K^{\text{para}}(\ell^{\text{val}_\ell(N)})$ for $\ell \neq p$ and $B_i \in \text{Sp}(4, \mathbb{Q})$ for $i \in \{1, \dots, t\}$. The space $S_k(K_{\mathcal{F}_1}, \chi_1)$ is contained in V , and moreover one verifies that the image of the restriction T of T_{χ_p} to $S_k(K_{\mathcal{F}_1}, \chi_1)$ is contained in $S_k(K_{\mathcal{F}_2}, \chi_2)$. By Lemma 3.3, the map $\mathcal{T} : S_k(\Gamma^{\text{para}}(N)) \rightarrow S_k(\Gamma^{\text{para}}(Np^4))$ given by (8) has the formula

$$\mathcal{T}(F) = \sum_{i=1}^t c_i \cdot F|_k B_i^{-1}, \quad F \in S_k(\Gamma^{\text{para}}(N)).$$

We let $\mathcal{T}_\chi = \mathcal{T}$; the formula in the statement of the theorem is a consequence of the expressions for c_i and B_i in Corollary 5.1.

Turning to the Hecke operators and the Atkin-Lehner operator, let $F \in S_k(\Gamma^{\text{para}}(N))$ and let $\Phi \in S_k(K_{\mathcal{F}_1}, \chi_1)$ with $F = F_\Phi$. Let ℓ be a rational prime with $\ell \neq p$ so that $\chi_{1,\ell}$ and $\chi_{2,\ell}$ are unramified. Moreover, we have that $\chi_{1,\ell}$ is trivial and $\chi_{2,\ell}(\ell) = \chi(\ell)$. Then, using (8) and Lemma 3.5 we have

$$\begin{aligned}
T(1, 1, \ell, \ell) \mathcal{T}_\chi(F_\Phi) &= T(1, 1, \ell, \ell) F_{T(\Phi)} \\
&= \ell^{k-3} \chi(\ell) F_{T_{1,0}(\ell) T(\Phi)}
\end{aligned}$$

$$\begin{aligned}
&= \ell^{k-3} \chi(\ell) F_{T(T_{1,0}(\ell)\Phi)} \\
&= \chi(\ell) \mathcal{T}_\chi(\ell^{k-3} F_{T_{1,0}(\ell)\Phi}) \\
&= \chi(\ell) \mathcal{T}_\chi(T(1, 1, \ell, \ell) F_\Phi).
\end{aligned}$$

The proofs for the other two operators are similar. \square

4. LOCAL CALCULATIONS

Throughout this section, we will use the following notation. Let F be a nonarchimedean local field of characteristic zero, with ring of integers \mathfrak{o} and generator ϖ of the maximal ideal \mathfrak{p} of \mathfrak{o} . We let q be the number of elements of $\mathfrak{o}/\mathfrak{p}$ and use the absolute value on F such that $|\varpi| = q^{-1}$. We use the Haar measure on the additive group F that assigns \mathfrak{o} measure 1 and the Haar measure on the multiplicative group F^\times that assigns \mathfrak{o}^\times measure $1 - q^{-1}$. We χ be a quadratic character of F^\times of conductor \mathfrak{p} . Let

$$J' = \begin{bmatrix} & & & 1 \\ & & 1 & \\ & -1 & & \\ -1 & & & \end{bmatrix}.$$

For this section only, we define $\mathrm{GSp}(4, F)$ as the subgroup of all $g \in \mathrm{GL}(4, F)$ such that ${}^t g J' g = \lambda(g) J'$ for some $\lambda(g) \in F^\times$ called the multiplier of g . For n a non-negative integer, we let $K(\mathfrak{p}^n)$ be the subgroup of $k \in \mathrm{GSp}(4, F)$ such that $\lambda(k) \in \mathfrak{o}^\times$ and

$$k \in \begin{bmatrix} \mathfrak{o} & \mathfrak{o} & \mathfrak{o} & \mathfrak{p}^{-n} \\ \mathfrak{p}^n & \mathfrak{o} & \mathfrak{o} & \mathfrak{o} \\ \mathfrak{p}^n & \mathfrak{o} & \mathfrak{o} & \mathfrak{o} \\ \mathfrak{p}^n & \mathfrak{p}^n & \mathfrak{p}^n & \mathfrak{o} \end{bmatrix}. \quad (13)$$

Throughout this section, (π, V) is a smooth representation of $\mathrm{GSp}(4, F)$ for which the center acts trivially. If n is a non-negative integer, then $V(n)$ is the subspace of vectors fixed by the paramodular subgroup $K(\mathfrak{p}^n)$; also, we let $V(n, \chi)$ be the subspace of vectors $v \in V$ such that $\pi(k)v = \chi(\lambda(k))v$ for $k \in K(\mathfrak{p}^n)$. Finally, let

$$\eta = \begin{bmatrix} \varpi^{-1} & & & \\ & 1 & & \\ & & 1 & \\ & & & \varpi \end{bmatrix}, \quad \tau = \begin{bmatrix} 1 & & & \\ & \varpi^{-1} & & \\ & & \varpi & \\ & & & 1 \end{bmatrix}, \quad t_4 = \begin{bmatrix} & & & -\varpi^{-4} \\ & & 1 & \\ & & & 1 \\ \varpi^4 & & & \end{bmatrix}.$$

Usually, we will write η and τ for $\pi(\eta)$ and $\pi(\tau)$, respectively.

In [JR2] we constructed a twisting map,

$$T_\chi : V(0) \rightarrow V(4, \chi), \quad (14)$$

given by

$$T_\chi(v) = q^3 \int_{\mathfrak{o}} \int_{\mathfrak{o}} \int_{\mathfrak{o}^\times} \int_{\mathfrak{o}^\times} \chi(ab) \pi \left(\begin{bmatrix} 1 & & & \\ & 1 & & \\ & x & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & -a\varpi^{-1} & b\varpi^{-2} & z\varpi^{-4} \\ & 1 & & b\varpi^{-2} \\ & & 1 & a\varpi^{-1} \\ & & & 1 \end{bmatrix} \right) \tau v \, da \, db \, dx \, dz \quad (\text{P1})$$

$$+ q^3 \int_{\mathfrak{o}} \int_{\mathfrak{p}} \int_{\mathfrak{o}^\times} \int_{\mathfrak{o}^\times} \chi(ab) \pi \left(\begin{bmatrix} 1 & & & \\ & -1 & 1 & \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & y & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & -a\varpi^{-1} & b\varpi^{-2} & z\varpi^{-4} \\ & 1 & & b\varpi^{-2} \\ & & 1 & a\varpi^{-1} \\ & & & 1 \end{bmatrix} \right) \tau v \, da \, db \, dy \, dz \quad (\text{P2})$$

$$+ q^2 \int_{\mathfrak{o}} \int_{\mathfrak{o}} \int_{\mathfrak{o}^\times} \int_{\mathfrak{o}^\times} \chi(ab) \pi(t_4 \begin{bmatrix} 1 & & & \\ & 1 & & \\ & x & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & -a\varpi^{-1} & b\varpi^{-2} & z\varpi^{-3} \\ & 1 & & b\varpi^{-2} \\ & & 1 & a\varpi^{-1} \\ & & & 1 \end{bmatrix}) \tau v \, da \, db \, dx \, dz \quad (\text{P3})$$

$$+ q^2 \int_{\mathfrak{o}} \int_{\mathfrak{p}} \int_{\mathfrak{o}^\times} \int_{\mathfrak{o}^\times} \chi(ab) \pi(t_4 \begin{bmatrix} 1 & & & \\ & -1 & 1 & \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & y & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & -a\varpi^{-1} & b\varpi^{-2} & z\varpi^{-3} \\ & 1 & & b\varpi^{-2} \\ & & 1 & a\varpi^{-1} \\ & & & 1 \end{bmatrix}) \tau v \, da \, db \, dy \, dz. \quad (\text{P4})$$

Remark 4.1. The Iwasawa decomposition asserts that $\mathrm{GSp}(4, F) = B \cdot \mathrm{GSp}(4, \mathfrak{o})$ where B is the Borel subgroup of upper-triangular matrices in $\mathrm{GSp}(4, F)$. Hence, if $v \in V(0)$ so that v is invariant under $\mathrm{GSp}(4, \mathfrak{o})$, then it is possible to obtain a formula for $T_\chi(v)$ involving only upper-triangular matrices. The remainder of this section will be devoted to calculating formulas for the terms (P1), (P2), (P3), (P4) involving only upper-triangular matrices. The resulting formula for $T_\chi(v)$ is given in the following theorem. The proof of this theorem follows from several technical lemmas. The full proofs of these lemmas are provided in an appendix to this paper [JR-a]. In some cases, we directly provide an Iwasawa identity $g = bk$ where $g \in \mathrm{GSp}(4, F)$, $b \in B$, and $k \in \mathrm{GSp}(4, \mathfrak{o})$. In many cases, we are able to obtain an appropriate Iwasawa identity by using the following formal matrix identity

$$\begin{bmatrix} 1 & \\ x & 1 \end{bmatrix} = \begin{bmatrix} 1 & x^{-1} \\ & 1 \end{bmatrix} \begin{bmatrix} -x^{-1} & \\ & -x \end{bmatrix} \begin{bmatrix} & 1 \\ -1 & \end{bmatrix} \begin{bmatrix} 1 & x^{-1} \\ & 1 \end{bmatrix}. \quad (15)$$

Both methods will require that we decompose the domains of integration in an advantageous manner. The assumptions on the character, $\chi \neq 1$, $\chi^2 = 1$ and $\chi(1 + \mathfrak{p}) = 1$, also play a significant role in the computations.

Theorem 4.2. *Let $v \in V(0)$. Then the twisting operator (14) is given by the formula*

$$T_\chi(v) = \sum_{k=1}^{14} T_\chi^k(v)$$

where

$$T_\chi^1(v) = q^2 \int_{\mathfrak{o}} \int_{\mathfrak{o}^\times} \int_{\mathfrak{o}^\times} \int_{\mathfrak{o}^\times} \chi(ab) \pi \left(\begin{bmatrix} 1 & -(a+xb)\varpi^{-1} & & \\ & 1 & & \\ & & 1 & (a+xb)\varpi^{-1} \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & b\varpi^{-2} & z\varpi^{-4} \\ & 1 & x^{-1}\varpi^{-1} & b\varpi^{-2} \\ & & 1 & \\ & & & 1 \end{bmatrix} \right) v \, da \, db \, dx \, dz,$$

$$\begin{aligned}
T_{\chi}^2(v) &= q\eta \int_{\mathfrak{o}^{\times} - (1+\mathfrak{p})} \int_{\mathfrak{o}^{\times}} \int_{\mathfrak{o}^{\times}} \int_{\mathfrak{o}^{\times} - (1+\mathfrak{p})} \chi(abxy) \pi \left(\begin{bmatrix} 1 & b\varpi^{-1} & & \\ & 1 & & \\ & & 1 & -b\varpi^{-1} \\ & & & 1 \end{bmatrix} \right. \\
&\quad \left. \begin{bmatrix} 1 & & -a\varpi^{-2} & ab(1 - (1-y)^{-1}x)\varpi^{-3} \\ & 1 & ab^{-1}(1-x)^{-1}\varpi^{-1} & -a\varpi^{-2} \\ & & 1 & \\ & & & 1 \end{bmatrix} \right) v \, dx \, da \, db \, dy, \\
T_{\chi}^3(v) &= \eta \int_{\mathfrak{o}^{\times} - (1+\mathfrak{p})} \int_{\mathfrak{o}^{\times}} \int_{\mathfrak{o}^{\times}} \chi(b(1-z)) \pi \left(\begin{bmatrix} 1 & a\varpi^{-2} & b\varpi^{-3} \\ & 1 & a^2b^{-1}z\varpi^{-1} & a\varpi^{-2} \\ & & 1 & \\ & & & 1 \end{bmatrix} \right) v \, da \, db \, dz, \\
T_{\chi}^4(v) &= q\eta \int_{\mathfrak{o}^{\times}} \int_{\mathfrak{o}^{\times}} \int_{\mathfrak{o}} \chi(b) \pi \left(\begin{bmatrix} 1 & x\varpi^{-2} & & \\ & 1 & & \\ & & 1 & -x\varpi^{-2} \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & a\varpi^{-2} & (b\varpi - ax)\varpi^{-4} \\ & 1 & a\varpi^{-2} \\ & & 1 & \\ & & & 1 \end{bmatrix} \right) v \, da \, db \, dx, \\
T_{\chi}^5(v) &= \eta \int_{\mathfrak{o}} \int_{\mathfrak{o}^{\times}} \int_{\mathfrak{o}^{\times}} \chi(b) \pi \left(\begin{bmatrix} 1 & x\varpi^{-1} & & \\ & 1 & & \\ & & 1 & -x\varpi^{-1} \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & a\varpi^{-2} & (b - ax)\varpi^{-3} \\ & 1 & a\varpi^{-2} \\ & & 1 & \\ & & & 1 \end{bmatrix} \right) v \, da \, db \, dx, \\
T_{\chi}^6(v) &= q^{-1}\eta^2 \int_{\mathfrak{o}^{\times}} \int_{\mathfrak{o}^{\times}} \int_{\mathfrak{o}^{\times}} \chi(bx) \pi \left(\begin{bmatrix} 1 & a\varpi^{-2} & b(1-x\varpi)\varpi^{-2} \\ & 1 & a^2b^{-1}\varpi^{-2} & a\varpi^{-2} \\ & & 1 & \\ & & & 1 \end{bmatrix} \right) v \, da \, db \, dx, \\
T_{\chi}^7(v) &= q\tau \int_{\mathfrak{o}} \int_{\mathfrak{o}^{\times}} \int_{\mathfrak{o}^{\times}} \chi(ab) \pi \left(\begin{bmatrix} 1 & -a\varpi^{-2} & & \\ & 1 & & \\ & & 1 & a\varpi^{-2} \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & b\varpi^{-1} & z\varpi^{-4} \\ & 1 & b\varpi^{-1} \\ & & 1 & \\ & & & 1 \end{bmatrix} \right) v \, da \, db \, dz, \\
T_{\chi}^8(v) &= \eta\tau \int_{\mathfrak{o}^{\times} - (1+\mathfrak{p})} \int_{\mathfrak{o}^{\times}} \int_{\mathfrak{o}^{\times}} \chi(abz(1-z)) \pi \left(\begin{bmatrix} 1 & b\varpi^{-2} & & \\ & 1 & & \\ & & 1 & -b\varpi^{-2} \\ & & & 1 \end{bmatrix} \right. \\
&\quad \left. \begin{bmatrix} 1 & a\varpi^{-1} & -ab(1-z)\varpi^{-3} \\ & 1 & a\varpi^{-1} \\ & & 1 & \\ & & & 1 \end{bmatrix} \right) v \, da \, db \, dz, \\
T_{\chi}^9(v) &= q^{-2}\eta^2\tau \int_{\mathfrak{o}^{\times}} \int_{\mathfrak{o}^{\times}} \int_{\mathfrak{o}^{\times}} \chi(b) \pi \left(\begin{bmatrix} 1 & a\varpi^{-1} & & \\ & 1 & & \\ & & 1 & -a\varpi^{-1} \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & -b\varpi^{-1} \\ & 1 & x\varpi^{-1} \\ & & 1 & \\ & & & 1 \end{bmatrix} \right) v \, da \, db \, dx,
\end{aligned}$$

$$\begin{aligned}
T_{\chi}^{10}(v) &= q^{-3} \eta^2 \tau^2 \int_{\mathfrak{o}^{\times}} \int_{\mathfrak{o}^{\times}} \chi(b) \pi \left(\begin{bmatrix} 1 & a\varpi^{-2} & -b\varpi^{-1} \\ & 1 & \\ & & 1 & -a\varpi^{-2} \\ & & & 1 \end{bmatrix} \right) v \, da \, db, \\
T_{\chi}^{11}(v) &= q^3 \tau^{-1} \int_{\mathfrak{o}} \int_{\mathfrak{o}^{\times}} \int_{\mathfrak{o}^{\times}} \int_{\mathfrak{o}^{\times}} \chi(ab) \pi \left(\begin{bmatrix} 1 & b\varpi^{-1} & & \\ & 1 & & \\ & & 1 & -b\varpi^{-1} \\ & & & 1 \end{bmatrix} \right. \\
&\quad \left. \begin{bmatrix} 1 & (xb+a\varpi)\varpi^{-3} & z\varpi^{-4} \\ & 1 & (xb+a\varpi)\varpi^{-3} \\ & & 1 & \\ & & & 1 \end{bmatrix} \right) v \, da \, db \, dx \, dz, \\
T_{\chi}^{12}(v) &= q^2 \eta \tau^{-1} \int_{\mathfrak{o}^{\times} - (1+\mathfrak{p})} \int_{\mathfrak{o}} \int_{\mathfrak{o}^{\times}} \int_{\mathfrak{o}^{\times}} \chi(abz(1-z)) \pi \left(\begin{bmatrix} 1 & a\varpi^{-1} & & \\ & 1 & & \\ & & 1 & -a\varpi^{-1} \\ & & & 1 \end{bmatrix} \right. \\
&\quad \left. \begin{bmatrix} 1 & y\varpi^{-3} & a(b(1-z)\varpi - y)\varpi^{-4} \\ & 1 & -a^{-1}(y+b\varpi)\varpi^{-2} & y\varpi^{-3} \\ & & 1 & \\ & & & 1 \end{bmatrix} \right) v \, da \, db \, dy \, dz, \\
T_{\chi}^{13}(v) &= \eta^2 \tau^{-1} \int_{\mathfrak{o}^{\times}} \int_{\mathfrak{o}^{\times}} \int_{\mathfrak{o}^{\times}} \chi(bx) \pi \left(\begin{bmatrix} 1 & a\varpi^{-2} & b(1-x)\varpi^{-1} \\ & 1 & a^2 b^{-1} \varpi^{-3} & a\varpi^{-2} \\ & & 1 & \\ & & & 1 \end{bmatrix} \right) v \, da \, db \, dx, \\
T_{\chi}^{14}(v) &= q \eta^2 \tau^{-2} \int_{\mathfrak{o}} \int_{\mathfrak{o}^{\times}} \int_{\mathfrak{o}^{\times}} \chi(b) \pi \left(\begin{bmatrix} 1 & a\varpi^{-2} & -b\varpi^{-1} \\ & 1 & x\varpi^{-4} & a\varpi^{-2} \\ & & 1 & \\ & & & 1 \end{bmatrix} \right) v \, da \, db \, dx.
\end{aligned}$$

Proof. Substituting the formulas from Lemmas 4.3, 4.4, 4.5 and 4.6 we have that the twisting operator is given by the formula

$$\begin{aligned}
T_{\chi}(v) &= \\
& q^2 \int_{\mathfrak{o}} \int_{\mathfrak{o}^{\times}} \int_{\mathfrak{o}^{\times}} \int_{\mathfrak{o}^{\times}} \chi(ab) \pi \left(\begin{bmatrix} 1 & -(a+xb)\varpi^{-1} & & \\ & 1 & & \\ & & 1 & (a+xb)\varpi^{-1} \\ & & & 1 \end{bmatrix} \right. \\
& \quad \left. \begin{bmatrix} 1 & b\varpi^{-2} & z\varpi^{-4} \\ & 1 & x^{-1}\varpi^{-1} & b\varpi^{-2} \\ & & 1 & \\ & & & 1 \end{bmatrix} \right) v \, da \, db \, dx \, dz \\
& + q \chi(-1) \eta \int_{\mathfrak{o}^{\times} - (1+\mathfrak{p})} \int_{\mathfrak{o}^{\times}} \int_{\mathfrak{o}^{\times}} \int_{\mathfrak{o}^{\times} - A(z)} \chi(abx) \pi \left(\begin{bmatrix} 1 & b\varpi^{-1} & & \\ & 1 & & \\ & & 1 & -b\varpi^{-1} \\ & & & 1 \end{bmatrix} \right)
\end{aligned}$$

$$\begin{aligned}
& \left[\begin{array}{ccc} 1 & a\varpi^{-2} & -abx^{-1}(1+x-z)\varpi^{-3} \\ & 1 & a\varpi^{-2} \\ & & 1 \end{array} \right] v \, dx \, da \, db \, dz \\
& + \chi(-1)\eta \int_{\mathfrak{o}^\times - (1+\mathfrak{p})} \int_{\mathfrak{o}^\times} \int_{\mathfrak{o}^\times} \chi(b(1-z))\pi \left(\begin{array}{ccc} 1 & a\varpi^{-2} & -b\varpi^{-3} \\ & 1 & a\varpi^{-2} \\ & & 1 \end{array} \right) v \, da \, db \, dz \\
& + q\eta \int_{\mathfrak{o}} \int_{\mathfrak{o}^\times} \int_{\mathfrak{o}^\times} \chi(b)\pi \left(\begin{array}{ccc} 1 & x\varpi^{-2} & \\ & 1 & -x\varpi^{-2} \\ & & 1 \end{array} \right) \left[\begin{array}{ccc} 1 & a\varpi^{-2} & (b\varpi - ax)\varpi^{-4} \\ & 1 & a\varpi^{-2} \\ & & 1 \end{array} \right] v \, da \, db \, dx \\
& + \eta \int_{\mathfrak{o}} \int_{\mathfrak{o}^\times} \int_{\mathfrak{o}^\times} \chi(ab)\pi \left(\begin{array}{ccc} 1 & -a\varpi^{-2} & \\ & 1 & a\varpi^{-2} \\ & & 1 \end{array} \right) \left[\begin{array}{ccc} 1 & y\varpi^{-1} & a(y+b)\varpi^{-3} \\ & 1 & y\varpi^{-1} \\ & & 1 \end{array} \right] v \, da \, db \, dy \\
& + q^{-1}\chi(-1)\eta^2 \int_{\mathfrak{o}^\times} \int_{\mathfrak{o}^\times} \int_{\mathfrak{o}^\times} \chi(bx)\pi \left(\begin{array}{ccc} 1 & a\varpi^{-2} & b(1+x\varpi)\varpi^{-2} \\ & 1 & a\varpi^{-2} \\ & & 1 \end{array} \right) v \, da \, db \, dx \\
& + q\tau \int_{\mathfrak{o}} \int_{\mathfrak{o}^\times} \int_{\mathfrak{o}^\times} \chi(ab)\pi \left(\begin{array}{ccc} 1 & -a\varpi^{-2} & \\ & 1 & a\varpi^{-2} \\ & & 1 \end{array} \right) \left[\begin{array}{ccc} 1 & b\varpi^{-1} & z\varpi^{-4} \\ & 1 & b\varpi^{-1} \\ & & 1 \end{array} \right] v \, da \, db \, dz \\
& + \eta\tau \int_{\mathfrak{o}^\times - (1+\mathfrak{p})} \int_{\mathfrak{o}^\times} \int_{\mathfrak{o}^\times} \chi(abz(1-z))\pi \left(\begin{array}{ccc} 1 & b\varpi^{-2} & \\ & 1 & \\ & & 1 \end{array} \right) \\
& \left[\begin{array}{ccc} 1 & a\varpi^{-1} & -ab(1-z)\varpi^{-3} \\ & 1 & a\varpi^{-1} \\ & & 1 \end{array} \right] v \, da \, db \, dz \\
& + q^{-2}\chi(-1)\eta^2\tau \int_{\mathfrak{o}^\times} \int_{\mathfrak{o}^\times} \int_{\mathfrak{o}^\times} \chi(b)\pi \left(\begin{array}{ccc} 1 & a\varpi^{-1} & \\ & 1 & -a\varpi^{-1} \\ & & 1 \end{array} \right) \left[\begin{array}{ccc} 1 & x\varpi^{-1} & b\varpi^{-1} \\ & 1 & \\ & & 1 \end{array} \right] v \, da \, db \, dx \\
& + q^{-3}\chi(-1)\eta^2\tau^2 \int_{\mathfrak{o}^\times} \int_{\mathfrak{o}^\times} \chi(b)\pi \left(\begin{array}{ccc} 1 & a\varpi^{-2} & b\varpi^{-1} \\ & 1 & \\ & & 1 \end{array} \right) v \, da \, db
\end{aligned}$$

$$\begin{aligned}
& + q^3 \tau^{-1} \int_{\mathfrak{o}} \int_{\mathfrak{o}^\times} \int_{\mathfrak{o}^\times} \int_{\mathfrak{o}^\times} \chi(ab) \pi \left(\begin{bmatrix} 1 & b\varpi^{-1} & & \\ & 1 & & \\ & & 1 & -b\varpi^{-1} \\ & & & 1 \end{bmatrix} \right. \\
& \left. \begin{bmatrix} 1 & & -(b-xa\varpi)x\varpi^{-3} & z\varpi^{-4} \\ & 1 & x\varpi^{-2} & -(b-xa\varpi)x\varpi^{-3} \\ & & 1 & \\ & & & 1 \end{bmatrix} \right) v \, da \, db \, dx \, dz \\
& + q^2 \tau^{-1} \int_{\mathfrak{o}} \int_{\mathfrak{o}} \int_{\mathfrak{o}^\times} \int_{\mathfrak{o}^\times} \chi(ab) \pi \left(\begin{bmatrix} 1 & b\varpi^{-1} & & \\ & 1 & & \\ & & 1 & -b\varpi^{-1} \\ & & & 1 \end{bmatrix} \right. \\
& \left. \begin{bmatrix} 1 & (a+by)\varpi^{-2} & z\varpi^{-4} \\ & 1 & -y\varpi^{-1} & (a+by)\varpi^{-2} \\ & & 1 & \\ & & & 1 \end{bmatrix} \right) v \, da \, db \, dy \, dz \\
& + q\eta\tau^{-1} \int_{\mathfrak{o}^\times - (1+\mathfrak{p})} \int_{\mathfrak{o}} \int_{\mathfrak{o}^\times} \int_{\mathfrak{o}^\times} \chi(ab) \pi \left(\begin{bmatrix} 1 & -a\varpi^{-1} & & \\ & 1 & & \\ & & 1 & a\varpi^{-1} \\ & & & 1 \end{bmatrix} \right. \\
& \left. \begin{bmatrix} 1 & & y\varpi^{-2} & a(y+bz)\varpi^{-3} \\ & 1 & a^{-1}(y+bz(z-1)^{-1})\varpi^{-1} & y\varpi^{-2} \\ & & 1 & \\ & & & 1 \end{bmatrix} \right) v \, da \, db \, dy \, dz \\
& + q^2 \chi(-1) \eta \tau^{-1} \int_{\mathfrak{o}^\times - (1+\mathfrak{p})} \int_{\mathfrak{o}^\times} \int_{\mathfrak{o}^\times} \int_{\mathfrak{o}^\times} \chi(ba(1-z)z) \pi \left(\begin{bmatrix} 1 & b\varpi^{-1} & & \\ & 1 & & \\ & & 1 & -b\varpi^{-1} \\ & & & 1 \end{bmatrix} \right. \\
& \left. \begin{bmatrix} 1 & & -x\varpi^{-3} & b(x-za\varpi)\varpi^{-4} \\ & 1 & b^{-1}(x+a\varpi)\varpi^{-2} & -x\varpi^{-3} \\ & & 1 & \\ & & & 1 \end{bmatrix} \right) v \, dx \, da \, db \, dz \\
& + \chi(-1) \eta^2 \tau^{-1} \int_{\mathfrak{o}^\times} \int_{\mathfrak{o}^\times} \int_{\mathfrak{o}^\times} \chi(bx) \pi \left(\begin{bmatrix} 1 & & a\varpi^{-2} & b(x-1)\varpi^{-1} \\ & 1 & -a^2 b^{-1} \varpi^{-3} & a\varpi^{-2} \\ & & 1 & \\ & & & 1 \end{bmatrix} \right) v \, da \, db \, dx \\
& + q\chi(-1) \eta^2 \tau^{-2} \int_{\mathfrak{o}^\times} \int_{\mathfrak{o}^\times} \int_{\mathfrak{o}^\times} \chi(b) \pi \left(\begin{bmatrix} 1 & & a\varpi^{-2} & b\varpi^{-1} \\ & 1 & x\varpi^{-4} & a\varpi^{-2} \\ & & 1 & \\ & & & 1 \end{bmatrix} \right) v \, da \, db \, dx
\end{aligned}$$

$$+ \eta^2 \tau^{-2} \int_{\mathfrak{o}} \int_{\mathfrak{o}^\times} \int_{\mathfrak{o}^\times} \chi(a) \pi \left(\begin{bmatrix} 1 & b\varpi^{-2} & -a\varpi^{-1} \\ & 1 & y\varpi^{-3} & b\varpi^{-2} \\ & & 1 & \\ & & & 1 \end{bmatrix} \right) v \, da \, db \, dy.$$

For the remainder of the proof, we will simplify by combining pairs of terms and rewriting certain domains. First we combine the terms involving $\eta^2 \tau^{-2}$.

$$\begin{aligned} & q\chi(-1)\eta^2\tau^{-2} \int_{\mathfrak{o}^\times} \int_{\mathfrak{o}^\times} \int_{\mathfrak{o}^\times} \chi(b) \pi \left(\begin{bmatrix} 1 & a\varpi^{-2} & b\varpi^{-1} \\ & 1 & x\varpi^{-4} & a\varpi^{-2} \\ & & 1 & \\ & & & 1 \end{bmatrix} \right) v \, da \, db \, dx \\ & + \eta^2\tau^{-2} \int_{\mathfrak{o}} \int_{\mathfrak{o}^\times} \int_{\mathfrak{o}^\times} \chi(a) \pi \left(\begin{bmatrix} 1 & b\varpi^{-2} & -a\varpi^{-1} \\ & 1 & y\varpi^{-3} & b\varpi^{-2} \\ & & 1 & \\ & & & 1 \end{bmatrix} \right) v \, da \, db \, dy \\ & = q\chi(-1)\eta^2\tau^{-2} \int_{\mathfrak{o}^\times} \int_{\mathfrak{o}^\times} \int_{\mathfrak{o}^\times} \chi(b) \pi \left(\begin{bmatrix} 1 & a\varpi^{-2} & b\varpi^{-1} \\ & 1 & x\varpi^{-4} & a\varpi^{-2} \\ & & 1 & \\ & & & 1 \end{bmatrix} \right) v \, da \, db \, dx \\ & + q\chi(-1)\eta^2\tau^{-2} \int_{\mathfrak{p}} \int_{\mathfrak{o}^\times} \int_{\mathfrak{o}^\times} \chi(a) \pi \left(\begin{bmatrix} 1 & b\varpi^{-2} & a\varpi^{-1} \\ & 1 & y\varpi^{-4} & b\varpi^{-2} \\ & & 1 & \\ & & & 1 \end{bmatrix} \right) v \, da \, db \, dy \\ & = q\chi(-1)\eta^2\tau^{-2} \int_{\mathfrak{o}} \int_{\mathfrak{o}^\times} \int_{\mathfrak{o}^\times} \chi(b) \pi \left(\begin{bmatrix} 1 & a\varpi^{-2} & b\varpi^{-1} \\ & 1 & x\varpi^{-4} & a\varpi^{-2} \\ & & 1 & \\ & & & 1 \end{bmatrix} \right) v \, da \, db \, dx. \end{aligned}$$

Similarly, we combine the terms involving $\eta\tau^{-1}$,

$$\begin{aligned} & q\eta\tau^{-1} \int_{\mathfrak{o}^\times - (1+\mathfrak{p})} \int_{\mathfrak{o}} \int_{\mathfrak{o}^\times} \int_{\mathfrak{o}^\times} \chi(ab) \pi \left(\begin{bmatrix} 1 & -a\varpi^{-1} & & \\ & 1 & & \\ & & 1 & a\varpi^{-1} \\ & & & 1 \end{bmatrix} \right) \\ & \left[\begin{bmatrix} 1 & & y\varpi^{-2} & a(y+bz)\varpi^{-3} \\ & 1 & a^{-1}(y+bz(z-1)^{-1})\varpi^{-1} & y\varpi^{-2} \\ & & 1 & \\ & & & 1 \end{bmatrix} \right) v \, da \, db \, dy \, dz \\ & + q^2\chi(-1)\eta\tau^{-1} \int_{\mathfrak{o}^\times - (1+\mathfrak{p})} \int_{\mathfrak{o}^\times} \int_{\mathfrak{o}^\times} \int_{\mathfrak{o}^\times} \chi(ba(1-z)z) \pi \left(\begin{bmatrix} 1 & b\varpi^{-1} & & \\ & 1 & & \\ & & 1 & -b\varpi^{-1} \\ & & & 1 \end{bmatrix} \right) \\ & \left[\begin{bmatrix} 1 & & -x\varpi^{-3} & b(x-za\varpi)\varpi^{-4} \\ & 1 & b^{-1}(x+a\varpi)\varpi^{-2} & -x\varpi^{-3} \\ & & 1 & \\ & & & 1 \end{bmatrix} \right) v \, dx \, da \, db \, dz \end{aligned}$$

$$= q^2 \chi(-1) \eta \tau^{-1} \int_{\mathfrak{o}^\times - (1+\mathfrak{p})} \int_{\mathfrak{o}} \int_{\mathfrak{o}^\times} \int_{\mathfrak{o}^\times} \chi(abz(1-z)) \pi \left(\begin{bmatrix} 1 & a\varpi^{-1} & & \\ & 1 & & \\ & & 1 & -a\varpi^{-1} \\ & & & 1 \end{bmatrix} \right. \\ \left. \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} y\varpi^{-3} & & & \\ & -a(y+b(1-z)\varpi)\varpi^{-4} & & \\ & & y\varpi^{-3} & \\ & & & 1 \end{bmatrix} \right) v da db dy dz,$$

the two terms involving τ^{-1} ,

$$q^3 \tau^{-1} \int_{\mathfrak{o}} \int_{\mathfrak{o}^\times} \int_{\mathfrak{o}^\times} \int_{\mathfrak{o}^\times} \chi(ab) \pi \left(\begin{bmatrix} 1 & b\varpi^{-1} & & \\ & 1 & & \\ & & 1 & -b\varpi^{-1} \\ & & & 1 \end{bmatrix} \right. \\ \left. \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} -(b-xa\varpi)x\varpi^{-3} & & & \\ & z\varpi^{-4} & & \\ & & -(b-xa\varpi)x\varpi^{-3} & \\ & & & 1 \end{bmatrix} \right) v da db dx dz \\ + q^2 \tau^{-1} \int_{\mathfrak{o}} \int_{\mathfrak{o}} \int_{\mathfrak{o}^\times} \int_{\mathfrak{o}^\times} \chi(ab) \pi \left(\begin{bmatrix} 1 & b\varpi^{-1} & & \\ & 1 & & \\ & & 1 & -b\varpi^{-1} \\ & & & 1 \end{bmatrix} \right. \\ \left. \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} (a+by)\varpi^{-2} & & & \\ & -y\varpi^{-1} & & \\ & & (a+by)\varpi^{-2} & \\ & & & 1 \end{bmatrix} \right) v da db dy dz \\ = q^3 \tau^{-1} \int_{\mathfrak{o}} \int_{\mathfrak{o}} \int_{\mathfrak{o}^\times} \int_{\mathfrak{o}^\times} \chi(ab) \pi \left(\begin{bmatrix} 1 & b\varpi^{-1} & & \\ & 1 & & \\ & & 1 & -b\varpi^{-1} \\ & & & 1 \end{bmatrix} \right. \\ \left. \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} (xb+a\varpi)\varpi^{-3} & & & \\ & -x\varpi^{-2} & & \\ & & (xb+a\varpi)\varpi^{-3} & \\ & & & 1 \end{bmatrix} \right) v da db dx dz.$$

and two of the terms that involve the η operator,

$$q\eta \int_{\mathfrak{o}} \int_{\mathfrak{o}^\times} \int_{\mathfrak{o}^\times} \chi(b) \pi \left(\begin{bmatrix} 1 & x\varpi^{-2} & & \\ & 1 & & \\ & & 1 & -x\varpi^{-2} \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} a\varpi^{-2} & (b\varpi-ax)\varpi^{-4} & & \\ & a\varpi^{-2} & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \right) v da db dx \\ + \eta \int_{\mathfrak{o}} \int_{\mathfrak{o}^\times} \int_{\mathfrak{o}^\times} \chi(ab) \pi \left(\begin{bmatrix} 1 & -a\varpi^{-2} & & \\ & 1 & & \\ & & 1 & a\varpi^{-2} \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} y\varpi^{-1} & a(y+b)\varpi^{-3} & & \\ & y\varpi^{-1} & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \right) v da db dy$$

$$\begin{aligned}
&= q\eta \int_{\mathfrak{o}^\times} \int_{\mathfrak{o}^\times} \int_{\mathfrak{o}} \chi(b) \pi \left(\begin{bmatrix} 1 & x\varpi^{-2} & & \\ & 1 & & \\ & & 1 & -x\varpi^{-2} \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & a\varpi^{-2} & (b\varpi - ax)\varpi^{-4} \\ & 1 & a\varpi^{-2} \\ & & 1 & \\ & & & 1 \end{bmatrix} \right) v \, da \, db \, dx \\
&+ \eta \int_{\mathfrak{o}} \int_{\mathfrak{o}^\times} \int_{\mathfrak{o}^\times} \chi(b) \pi \left(\begin{bmatrix} 1 & x\varpi^{-1} & & \\ & 1 & & \\ & & 1 & -x\varpi^{-1} \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & a\varpi^{-2} & (b - ax)\varpi^{-3} \\ & 1 & a\varpi^{-2} \\ & & 1 & \\ & & & 1 \end{bmatrix} \right) v \, da \, db \, dx.
\end{aligned}$$

We rewrite one of the terms involving η after making the observation that if $z \in \mathfrak{o}^\times - (1 + \mathfrak{p})$ and f is a locally constant function on \mathfrak{o}^\times , then

$$\int_{\mathfrak{o}^\times - A(z)} f(x) dx = \int_{\mathfrak{o}^\times - (1 + \mathfrak{p})} f((z^{-1} - 1)(w^{-1} - 1)^{-1}) dw.$$

Hence

$$\begin{aligned}
&q\chi(-1)\eta \int_{\mathfrak{o}^\times - (1 + \mathfrak{p})} \int_{\mathfrak{o}^\times} \int_{\mathfrak{o}^\times} \int_{\mathfrak{o}^\times - A(z)} \chi(abx) \pi \left(\begin{bmatrix} 1 & b\varpi^{-1} & & \\ & 1 & & \\ & & 1 & -b\varpi^{-1} \\ & & & 1 \end{bmatrix} \right. \\
&\quad \left. \begin{bmatrix} 1 & & a\varpi^{-2} & -abx^{-1}(1 + x - z)\varpi^{-3} \\ & 1 & -ab^{-1}xz(1 - z + zx)^{-1}\varpi^{-1} & a\varpi^{-2} \\ & & 1 & \\ & & & 1 \end{bmatrix} \right) v \, dx \, da \, db \, dz \\
&= q\chi(-1)\eta \int_{\mathfrak{o}^\times - (1 + \mathfrak{p})} \int_{\mathfrak{o}^\times} \int_{\mathfrak{o}^\times} \int_{\mathfrak{o}^\times - (1 + \mathfrak{p})} \chi(abz(1 - z)w(1 - w)) \pi \left(\begin{bmatrix} 1 & b\varpi^{-1} & & \\ & 1 & & \\ & & 1 & -b\varpi^{-1} \\ & & & 1 \end{bmatrix} \right. \\
&\quad \left. \begin{bmatrix} 1 & & a\varpi^{-2} & -ab(1 + zw^{-1}(1 - w))\varpi^{-3} \\ & 1 & -ab^{-1}w\varpi^{-1} & a\varpi^{-2} \\ & & 1 & \\ & & & 1 \end{bmatrix} \right) v \, dw \, da \, db \, dz \\
&= q\chi(-1)\eta \int_{\mathfrak{o}^\times - (1 + \mathfrak{p})} \int_{\mathfrak{o}^\times} \int_{\mathfrak{o}^\times} \int_{\mathfrak{o}^\times - (1 + \mathfrak{p})} \chi(az^{-1}(1 - z)bw^{-1}(1 - w)) \pi \left(\begin{bmatrix} 1 & b\varpi^{-1} & & \\ & 1 & & \\ & & 1 & -b\varpi^{-1} \\ & & & 1 \end{bmatrix} \right. \\
&\quad \left. \begin{bmatrix} 1 & & a\varpi^{-2} & -ab(1 + zw^{-1}(1 - w))\varpi^{-3} \\ & 1 & -ab^{-1}w\varpi^{-1} & a\varpi^{-2} \\ & & 1 & \\ & & & 1 \end{bmatrix} \right) v \, dw \, da \, db \, dz \\
&= q\chi(-1)\eta \int_{\mathfrak{o}^\times - (1 + \mathfrak{p})} \int_{\mathfrak{o}^\times} \int_{\mathfrak{o}^\times} \int_{\mathfrak{o}^\times - (1 + \mathfrak{p})} \chi(a(z^{-1} - 1)b(w^{-1} - 1)) \pi \left(\begin{bmatrix} 1 & b\varpi^{-1} & & \\ & 1 & & \\ & & 1 & -b\varpi^{-1} \\ & & & 1 \end{bmatrix} \right.
\end{aligned}$$

$$\begin{aligned}
& \begin{bmatrix} 1 & a\varpi^{-2} & -ab(1+z(w^{-1}-1))\varpi^{-3} \\ & 1 & -ab^{-1}w\varpi^{-1} & a\varpi^{-2} \\ & & 1 & \\ & & & 1 \end{bmatrix} v dw da db dz \\
&= q\chi(-1)\eta \int_{\mathfrak{o}^\times - (1+\mathfrak{p})} \int_{\mathfrak{o}^\times} \int_{\mathfrak{o}^\times} \int_{\mathfrak{o}^\times - (1+\mathfrak{p})} \chi(a(z-1)b(w-1))\pi\left(\begin{bmatrix} 1 & b\varpi^{-1} & & \\ & 1 & & \\ & & 1 & -b\varpi^{-1} \\ & & & 1 \end{bmatrix}\right) \\
& \begin{bmatrix} 1 & a\varpi^{-2} & -ab(1+z^{-1}(w-1))\varpi^{-3} \\ & 1 & -ab^{-1}w^{-1}\varpi^{-1} & a\varpi^{-2} \\ & & 1 & \\ & & & 1 \end{bmatrix} v dw da db dz \\
&= q\chi(-1)\eta \int_{\mathfrak{o}^\times - (-1+\mathfrak{p})} \int_{\mathfrak{o}^\times} \int_{\mathfrak{o}^\times} \int_{\mathfrak{o}^\times - (-1+\mathfrak{p})} \chi(aybx)\pi\left(\begin{bmatrix} 1 & b\varpi^{-1} & & \\ & 1 & & \\ & & 1 & -b\varpi^{-1} \\ & & & 1 \end{bmatrix}\right) \\
& \begin{bmatrix} 1 & a\varpi^{-2} & -ab(1+(1+y)^{-1}x)\varpi^{-3} \\ & 1 & -ab^{-1}(1+x)^{-1}\varpi^{-1} & a\varpi^{-2} \\ & & 1 & \\ & & & 1 \end{bmatrix} v dx da db dy \\
&= q\eta \int_{\mathfrak{o}^\times - (1+\mathfrak{p})} \int_{\mathfrak{o}^\times} \int_{\mathfrak{o}^\times} \int_{\mathfrak{o}^\times - (1+\mathfrak{p})} \chi(abxy)\pi\left(\begin{bmatrix} 1 & b\varpi^{-1} & & \\ & 1 & & \\ & & 1 & -b\varpi^{-1} \\ & & & 1 \end{bmatrix}\right) \\
& \begin{bmatrix} 1 & -a\varpi^{-2} & ab(1-(1-y)^{-1}x)\varpi^{-3} \\ & 1 & ab^{-1}(1-x)^{-1}\varpi^{-1} & -a\varpi^{-2} \\ & & 1 & \\ & & & 1 \end{bmatrix} v dx da db dy.
\end{aligned}$$

Finally, we are able to eliminate the factor $\chi(-1)$ from all terms using an appropriate change of variables. Substituting the simplified terms into the formula for $T_\chi(v)$, we obtain the result. \square

The proofs of the following four lemmas are provided in an appendix to this paper, [JR-a].

Lemma 4.3. *If $v \in V(0)$, then we have that (P1) is given by*

$$\begin{aligned}
& q^3 \int_{\mathfrak{o}} \int_{\mathfrak{o}} \int_{\mathfrak{o}^\times} \int_{\mathfrak{o}^\times} \chi(ab)\pi\left(\begin{bmatrix} 1 & & & \\ & 1 & & \\ & x & 1 & \\ & & & 1 \end{bmatrix}\right) \begin{bmatrix} 1 & -a\varpi^{-1} & b\varpi^{-2} & z\varpi^{-4} \\ & 1 & & b\varpi^{-2} \\ & & 1 & a\varpi^{-1} \\ & & & 1 \end{bmatrix} \tau v da db dx dz \\
&= q\tau \int_{\mathfrak{o}} \int_{\mathfrak{o}^\times} \int_{\mathfrak{o}^\times} \chi(ab)\pi\left(\begin{bmatrix} 1 & -a\varpi^{-2} & & \\ & 1 & & \\ & & 1 & a\varpi^{-2} \\ & & & 1 \end{bmatrix}\right) \begin{bmatrix} 1 & b\varpi^{-1} & z\varpi^{-4} \\ & 1 & b\varpi^{-1} \\ & & 1 \\ & & & 1 \end{bmatrix} v da db dz
\end{aligned}$$

$$\begin{aligned}
& + q^3 \tau^{-1} \int_{\mathfrak{o}} \int_{\mathfrak{o}^\times} \int_{\mathfrak{o}^\times} \int_{\mathfrak{o}^\times} \chi(ab) \pi \left(\begin{bmatrix} 1 & b\varpi^{-1} & & \\ & 1 & & \\ & & 1 & -b\varpi^{-1} \\ & & & 1 \end{bmatrix} \right. \\
& \left. \begin{bmatrix} 1 & -(b-xa\varpi)x\varpi^{-3} & z\varpi^{-4} \\ & x\varpi^{-2} & -(b-xa\varpi)x\varpi^{-3} \\ & & 1 \\ & & & 1 \end{bmatrix} \right) v \, da \, db \, dx \, dz \\
& + q^2 \int_{\mathfrak{o}} \int_{\mathfrak{o}^\times} \int_{\mathfrak{o}^\times} \int_{\mathfrak{o}^\times} \chi(ab) \pi \left(\begin{bmatrix} 1 & -(a+xb)\varpi^{-1} & & \\ & 1 & & \\ & & 1 & (a+xb)\varpi^{-1} \\ & & & 1 \end{bmatrix} \right. \\
& \left. \begin{bmatrix} 1 & b\varpi^{-2} & z\varpi^{-4} \\ & 1 & x^{-1}\varpi^{-1} & b\varpi^{-2} \\ & & 1 \\ & & & 1 \end{bmatrix} \right) v \, da \, db \, dx \, dz.
\end{aligned}$$

Proof. See the appendix [JR-a]. □

Lemma 4.4. *If $v \in V(0)$, then we have that (P2) is given by*

$$\begin{aligned}
& q^3 \int_{\mathfrak{o}} \int_{\mathfrak{p}} \int_{\mathfrak{o}^\times} \int_{\mathfrak{o}^\times} \chi(ab) \pi \left(\begin{bmatrix} 1 & & & \\ & 1 & & \\ & -1 & & \\ & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & y & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & -a\varpi^{-1} & b\varpi^{-2} & z\varpi^{-4} \\ & 1 & & b\varpi^{-2} \\ & & 1 & a\varpi^{-1} \\ & & & 1 \end{bmatrix} \right) \tau v \, da \, db \, dy \, dz \\
& = q^2 \tau^{-1} \int_{\mathfrak{o}} \int_{\mathfrak{o}} \int_{\mathfrak{o}^\times} \int_{\mathfrak{o}^\times} \chi(ab) \pi \left(\begin{bmatrix} 1 & b\varpi^{-1} & & \\ & 1 & & \\ & & 1 & -b\varpi^{-1} \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & (a+by)\varpi^{-2} & z\varpi^{-4} \\ & 1 & -y\varpi^{-1} & (a+by)\varpi^{-2} \\ & & 1 \\ & & & 1 \end{bmatrix} \right) v \, da \, db \, dy \, dz.
\end{aligned}$$

Proof. See the appendix [JR-a]. □

Lemma 4.5. *If $v \in V(0)$, then we have that (P3) is given by*

$$\begin{aligned}
& q^2 \int_{\mathfrak{o}} \int_{\mathfrak{o}} \int_{\mathfrak{o}^\times} \int_{\mathfrak{o}^\times} \chi(ab) \pi(t_4 \begin{bmatrix} 1 & & & \\ & 1 & & \\ & x & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & -a\varpi^{-1} & b\varpi^{-2} & z\varpi^{-3} \\ & 1 & & b\varpi^{-2} \\ & & 1 & a\varpi^{-1} \\ & & & 1 \end{bmatrix}) \tau v \, da \, db \, dx \, dz \\
& = q^2 \chi(-1) \int_{\mathfrak{o}^\times - (1+\mathfrak{p})} \int_{\mathfrak{o}^\times} \int_{\mathfrak{o}^\times} \int_{\mathfrak{o}^\times} \chi(abx(1-z)) \eta \tau^{-1} \pi \left(\begin{bmatrix} 1 & b\varpi^{-1} & & \\ & 1 & & \\ & & 1 & -b\varpi^{-1} \\ & & & 1 \end{bmatrix} \right. \\
& \left. \begin{bmatrix} 1 & a\varpi^{-3} & -ab(1+x\varpi)\varpi^{-4} \\ & 1 & -ab^{-1}(1+xz^{-1}\varpi)^{-1}\varpi^{-2} & a\varpi^{-3} \\ & & 1 \\ & & & 1 \end{bmatrix} \right) v \, dx \, da \, db \, dz
\end{aligned}$$

$$\begin{aligned}
& + \int_{\mathfrak{o}^\times - (1+\mathfrak{p})} \int_{\mathfrak{o}^\times} \int_{\mathfrak{o}^\times} \int_{\mathfrak{o}} \chi(abz(1-z)) \eta \tau \pi \left(\begin{bmatrix} 1 & b\varpi^{-2} & & \\ & 1 & & \\ & & 1 & -b\varpi^{-2} \\ & & & 1 \end{bmatrix} \right. \\
& \left. \begin{bmatrix} 1 & a\varpi^{-1} & -ab(1-z)\varpi^{-3} \\ & 1 & a\varpi^{-1} \\ & & 1 \\ & & & 1 \end{bmatrix} \right) v \, dx \, da \, db \, dz \\
& + q\chi(-1) \int_{\mathfrak{o}^\times - (1+\mathfrak{p})} \int_{\mathfrak{o}^\times} \int_{\mathfrak{o}^\times} \int_{\mathfrak{o}^\times - A(z)} \chi(abx) \eta \pi \left(\begin{bmatrix} 1 & b\varpi^{-1} & & \\ & 1 & & \\ & & 1 & -b\varpi^{-1} \\ & & & 1 \end{bmatrix} \right. \\
& \left. \begin{bmatrix} 1 & a\varpi^{-2} & abx^{-1}(1+x-z)\varpi^{-3} \\ & 1 & -ab^{-1}xz(1-z+zx)^{-1}\varpi^{-1} & a\varpi^{-2} \\ & & 1 \\ & & & 1 \end{bmatrix} \right) v \, dx \, da \, db \, dz \\
& + \chi(-1) \int_{\mathfrak{o}^\times - (1+\mathfrak{p})} \int_{\mathfrak{o}^\times} \int_{\mathfrak{o}^\times} \int_{\mathfrak{o}} \chi(b(1-z)) \eta \pi \left(\begin{bmatrix} 1 & a\varpi^{-2} & -b\varpi^{-3} \\ & 1 & -a^2b^{-1}z\varpi^{-1} & a\varpi^{-2} \\ & & 1 \\ & & & 1 \end{bmatrix} \right) v \, dx \, da \, db \, dz \\
& + q\chi(-1) \int_{\mathfrak{o}^\times} \int_{\mathfrak{o}^\times} \int_{\mathfrak{o}^\times} \chi(b) \eta^2 \tau^{-2} \pi \left(\begin{bmatrix} 1 & a\varpi^{-2} & b\varpi^{-1} \\ & 1 & x\varpi^{-4} & a\varpi^{-2} \\ & & 1 \\ & & & 1 \end{bmatrix} \right) v \, da \, db \, dx \\
& + q^{-1}\chi(-1) \int_{\mathfrak{o}^\times} \int_{\mathfrak{o}^\times} \int_{\mathfrak{o}^\times} \chi(bx) \eta^2 \pi \left(\begin{bmatrix} 1 & a\varpi^{-2} & b(1+x\varpi)\varpi^{-2} \\ & 1 & a^2b^{-1}\varpi^{-2} & a\varpi^{-2} \\ & & 1 \\ & & & 1 \end{bmatrix} \right) v \, da \, db \, dx \\
& + q^{-2}\chi(-1) \int_{\mathfrak{o}^\times} \int_{\mathfrak{o}^\times} \int_{\mathfrak{o}^\times} \chi(b) \eta^2 \tau \pi \left(\begin{bmatrix} 1 & a\varpi^{-1} & & \\ & 1 & & \\ & & 1 & -a\varpi^{-1} \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & b\varpi^{-1} \\ & 1 & x\varpi^{-1} \\ & & 1 \\ & & & 1 \end{bmatrix} \right) v \, da \, db \, dx \\
& + q^{-3}\chi(-1) \int_{\mathfrak{o}^\times} \int_{\mathfrak{o}^\times} \chi(b) \eta^2 \tau^2 \pi \left(\begin{bmatrix} 1 & a\varpi^{-2} & b\varpi^{-1} \\ & 1 & & \\ & & 1 & -a\varpi^{-2} \\ & & & 1 \end{bmatrix} \right) v \, da \, db \\
& + q \int_{\mathfrak{o}^\times} \int_{\mathfrak{o}^\times} \int_{\mathfrak{o}^\times} \chi(b) \eta \pi \left(\begin{bmatrix} 1 & x\varpi^{-2} & & \\ & 1 & & \\ & & 1 & -x\varpi^{-2} \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & a\varpi^{-2} & (b\varpi - ax)\varpi^{-4} \\ & 1 & a\varpi^{-2} \\ & & 1 \\ & & & 1 \end{bmatrix} \right) v \, da \, db \, dx.
\end{aligned}$$

Proof. See the appendix [JR-a].

□

Lemma 4.6. *If $v \in V(0)$, then we have that (P4) is given by*

$$\begin{aligned}
& q^2 \int_{\mathfrak{o}} \int_{\mathfrak{p}} \int_{\mathfrak{o}^\times} \int_{\mathfrak{o}^\times} \chi(ab) \pi(t_4 \begin{bmatrix} 1 & & & \\ & -1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & y & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & -a\varpi^{-1} & b\varpi^{-2} & z\varpi^{-3} \\ & 1 & & b\varpi^{-2} \\ & & 1 & a\varpi^{-1} \\ & & & 1 \end{bmatrix}) \tau v \, da \, db \, dy \, dz \\
&= \int_{\mathfrak{o}} \int_{\mathfrak{o}^\times} \int_{\mathfrak{o}^\times} \chi(ab) \eta \pi \left(\begin{bmatrix} 1 & -a\varpi^{-2} & & \\ & 1 & & \\ & & 1 & a\varpi^{-2} \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & y\varpi^{-1} & a(y+b)\varpi^{-3} \\ & 1 & & y\varpi^{-1} \\ & & 1 & \\ & & & 1 \end{bmatrix} \right) v \, da \, db \, dy \\
&+ q \int_{\mathfrak{o}^\times - (1+\mathfrak{p})} \int_{\mathfrak{o}} \int_{\mathfrak{o}^\times} \int_{\mathfrak{o}^\times} \chi(ab) \eta \tau^{-1} \pi \left(\begin{bmatrix} 1 & -a\varpi^{-1} & & \\ & 1 & & \\ & & 1 & a\varpi^{-1} \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & y\varpi^{-2} & a(y+bz)\varpi^{-3} \\ & 1 & & y\varpi^{-2} \\ & & 1 & \\ & & & 1 \end{bmatrix} \right) v \, da \, db \, dy \, dz \\
&+ \int_{\mathfrak{o}} \int_{\mathfrak{o}^\times} \int_{\mathfrak{o}^\times} \chi(a) \eta^2 \tau^{-2} \pi \left(\begin{bmatrix} 1 & b\varpi^{-2} & -a\varpi^{-1} \\ & 1 & y\varpi^{-3} & b\varpi^{-2} \\ & & 1 & \\ & & & 1 \end{bmatrix} \right) v \, da \, db \, dy.
\end{aligned}$$

Proof. See the appendix [JR-a]. □

5. FORMULAS FOR \mathbb{Q}_p

In this section we will write the formula for the twisting operator (14) in the case where $F = \mathbb{Q}_p$ and $\mathrm{GSp}(4)$ is written with respect to J , as given in the introduction. We use the isomorphism between $\mathrm{GSp}(4)$ as defined with respect to J and $\mathrm{GSp}(4)$ as defined with respect to J' , as in Section 4, given in both directions by conjugation by the matrix

$$C = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}.$$

Under this isomorphism, the paramodular group $K(p^n)$ of level p^n as defined in (13) with respect to the J' realization, as in Section 4, is mapped to $K^{\mathrm{para}}(p^n)$. Let

$$\eta' = \begin{bmatrix} p^{-1} & & & \\ & 1 & & \\ & & p & \\ & & & 1 \end{bmatrix} \quad \text{and} \quad \tau' = \begin{bmatrix} 1 & & & \\ & p^{-1} & & \\ & & 1 & \\ & & & p \end{bmatrix}.$$

Corollary 5.1. *Let p be a prime of \mathbb{Q} . Let (π, V) be a smooth representation of $\mathrm{GSp}(4, \mathbb{Q}_p)$ for which the center acts trivially, and let χ be a quadratic character of \mathbb{Q}_p^\times of conductor p . Let $v \in V$ be such that $\pi(k)v = v$ for $k \in \mathrm{GSp}(4, \mathbb{Z}_p)$. Then, the twisting map $T_\chi : V(0) \rightarrow V(4, \chi)$ is given*

by the formula $T_\chi(v) = \sum_{i=1}^{14} T_\chi^i(v)$ for $v \in V(0)$, where

$$T_\chi^1(v) = p^{-11} \sum_{\substack{a,b,x \in (\mathbb{Z}_p/p^3\mathbb{Z}_p)^\times \\ z \in \mathbb{Z}_p/p^4\mathbb{Z}_p}} \chi(ab)\pi\left(\begin{bmatrix} 1 & -(a+xb)p^{-1} & & \\ & 1 & & \\ & & 1 & \\ & & (a+xb)p^{-1} & 1 \end{bmatrix} \begin{bmatrix} 1 & & zp^{-4} & bp^{-2} \\ & 1 & bp^{-2} & x^{-1}p^{-1} \\ & & 1 & \\ & & & 1 \end{bmatrix}\right)v,$$

$$T_\chi^2(v) = p^{-11}\eta' \sum_{\substack{a,b \in (\mathbb{Z}_p/p^3\mathbb{Z}_p)^\times \\ x,y \in (\mathbb{Z}_p/p^3\mathbb{Z}_p)^\times \\ x,y \neq 1(p)}} \chi(abxy)\pi\left(\begin{bmatrix} 1 & bp^{-1} & & \\ & 1 & & \\ & & 1 & \\ & & -bp^{-1} & 1 \end{bmatrix} \begin{bmatrix} 1 & ab(1-(1-y)^{-1}x)p^{-3} & -ap^{-2} \\ & 1 & -ap^{-2} \\ & & 1 & \\ & & & 1 \end{bmatrix}\right)v,$$

$$T_\chi^3(v) = p^{-6}\eta' \sum_{\substack{a \in (\mathbb{Z}_p/p^2\mathbb{Z}_p)^\times \\ b \in (\mathbb{Z}_p/p^3\mathbb{Z}_p)^\times \\ z \in (\mathbb{Z}_p/p\mathbb{Z}_p)^\times \\ z \neq 1(p)}} \chi(b(1-z))\pi\left(\begin{bmatrix} 1 & & bp^{-3} & ap^{-2} \\ & 1 & ap^{-2} & a^2b^{-1}zp^{-1} \\ & & 1 & \\ & & & 1 \end{bmatrix}\right)v,$$

$$T_\chi^4(v) = p^{-10}\eta' \sum_{\substack{a \in \mathbb{Z}_p/p^4\mathbb{Z}_p \\ b \in (\mathbb{Z}_p/p^3\mathbb{Z}_p)^\times \\ x \in (\mathbb{Z}_p/p^4\mathbb{Z}_p)^\times}} \chi(b)\pi\left(\begin{bmatrix} 1 & xp^{-2} & & \\ & 1 & & \\ & & 1 & \\ & & -xp^{-2} & 1 \end{bmatrix} \begin{bmatrix} 1 & & (bp-ax)p^{-4} & ap^{-2} \\ & 1 & ap^{-2} & \\ & & 1 & \\ & & & 1 \end{bmatrix}\right)v,$$

$$T_\chi^5(v) = p^{-9}\eta' \sum_{\substack{a,b \in (\mathbb{Z}_p/p^3\mathbb{Z}_p)^\times \\ x \in \mathbb{Z}_p/p^3\mathbb{Z}_p}} \chi(b)\pi\left(\begin{bmatrix} 1 & xp^{-1} & & \\ & 1 & & \\ & & 1 & \\ & & -xp^{-1} & 1 \end{bmatrix} \begin{bmatrix} 1 & & (b-ax)p^{-3} & ap^{-2} \\ & 1 & ap^{-2} & \\ & & 1 & \\ & & & 1 \end{bmatrix}\right)v,$$

$$T_\chi^6(v) = p^{-6}\eta'^2 \sum_{\substack{a,b \in (\mathbb{Z}_p/p^2\mathbb{Z}_p)^\times \\ x \in (\mathbb{Z}_p/p\mathbb{Z}_p)^\times}} \chi(bx)\pi\left(\begin{bmatrix} 1 & & b(1-xp)p^{-2} & ap^{-2} \\ & 1 & ap^{-2} & a^2b^{-1}p^{-2} \\ & & 1 & \\ & & & 1 \end{bmatrix}\right)v,$$

$$T_\chi^7(v) = p^{-7}\tau' \sum_{\substack{a \in (\mathbb{Z}_p/p^3\mathbb{Z}_p)^\times \\ b \in (\mathbb{Z}_p/p\mathbb{Z}_p)^\times \\ z \in \mathbb{Z}_p/p^4\mathbb{Z}_p}} \chi(ab)\pi\left(\begin{bmatrix} 1 & -ap^{-2} & & \\ & 1 & & \\ & & 1 & \\ & & ap^{-2} & 1 \end{bmatrix} \begin{bmatrix} 1 & & zp^{-4} & bp^{-1} \\ & 1 & bp^{-1} & \\ & & 1 & \\ & & & 1 \end{bmatrix}\right)v,$$

$$T_\chi^8(v) = p^{-9}\eta'\tau' \sum_{\substack{a,b,z \in (\mathbb{Z}_p/p^3\mathbb{Z}_p)^\times \\ z \neq 1(p)}} \chi(abz(1-z))\pi\left(\begin{bmatrix} 1 & bp^{-2} & & \\ & 1 & & \\ & & 1 & \\ & & -bp^{-2} & 1 \end{bmatrix}\right)v,$$

$$\begin{aligned}
& \begin{bmatrix} 1 & -ab(1-z)p^{-3} & ap^{-1} \\ & 1 & ap^{-1} \\ & & 1 \\ & & & 1 \end{bmatrix} v, \\
T_\chi^9(v) &= p^{-6} \eta'^2 \tau' \sum_{\substack{a \in (\mathbb{Z}_p/p^2\mathbb{Z}_p)^\times \\ b, x \in (\mathbb{Z}_p/p\mathbb{Z}_p)^\times}} \chi(b) \pi \left(\begin{bmatrix} 1 & ap^{-1} & & \\ & 1 & & \\ & & 1 & \\ & & -ap^{-1} & 1 \end{bmatrix} \begin{bmatrix} 1 & -bp^{-1} & xp^{-1} \\ & 1 & \\ & & 1 \\ & & & 1 \end{bmatrix} \right) v, \\
T_\chi^{10}(v) &= p^{-6} \eta'^2 \tau'^2 \sum_{\substack{a \in (\mathbb{Z}_p/p^2\mathbb{Z}_p)^\times \\ b \in (\mathbb{Z}_p/p\mathbb{Z}_p)^\times}} \chi(b) \pi \left(\begin{bmatrix} 1 & ap^{-2} & -bp^{-1} \\ & 1 & \\ & & 1 \\ & & -ap^{-2} & 1 \end{bmatrix} \right) v, \\
T_\chi^{11}(v) &= p^{-10} \tau'^{-1} \sum_{\substack{a \in (\mathbb{Z}_p/p^2\mathbb{Z}_p)^\times \\ b \in (\mathbb{Z}_p/p^4\mathbb{Z}_p)^\times \\ x \in \mathbb{Z}_p/p^3\mathbb{Z}_p \\ z \in \mathbb{Z}_p/p^4\mathbb{Z}_p}} \chi(ab) \pi \left(\begin{bmatrix} 1 & bp^{-1} & & \\ & 1 & & \\ & & 1 & \\ & & -bp^{-1} & 1 \end{bmatrix} \right. \\
& \quad \left. \begin{bmatrix} 1 & zp^{-4} & (xb+ap)p^{-3} \\ & 1 & (xb+ap)p^{-3} \\ & & 1 \\ & & & 1 \end{bmatrix} \right) v, \\
T_\chi^{12}(v) &= p^{-12} \eta' \tau'^{-1} \sum_{\substack{y \in \mathbb{Z}_p/p^4\mathbb{Z}_p \\ a \in (\mathbb{Z}_p/p^4\mathbb{Z}_p)^\times \\ b, z \in (\mathbb{Z}_p/p^3\mathbb{Z}_p)^\times \\ z \neq 1(p)}} \chi(abz(1-z)) \pi \left(\begin{bmatrix} 1 & ap^{-1} & & \\ & 1 & & \\ & & 1 & \\ & & -ap^{-1} & 1 \end{bmatrix} \right. \\
& \quad \left. \begin{bmatrix} 1 & a(b(1-z)p-y)p^{-4} & yp^{-3} \\ & 1 & yp^{-3} \\ & & 1 \\ & & & 1 \end{bmatrix} \right) v, \\
T_\chi^{13}(v) &= p^{-6} \eta'^2 \tau'^{-1} \sum_{\substack{a \in (\mathbb{Z}_p/p^2\mathbb{Z}_p)^\times \\ b \in (\mathbb{Z}_p/p^3\mathbb{Z}_p)^\times \\ x \in (\mathbb{Z}_p/p\mathbb{Z}_p)^\times}} \chi(bx) \pi \left(\begin{bmatrix} 1 & b(1-x)p^{-1} & ap^{-2} \\ & 1 & ap^{-2} \\ & & 1 \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & -bp^{-1} & ap^{-2} \\ & 1 & xp^{-4} \\ & & 1 \\ & & & 1 \end{bmatrix} \right) v, \\
T_\chi^{14}(v) &= p^{-6} \eta'^2 \tau'^{-2} \sum_{\substack{a \in (\mathbb{Z}_p/p^2\mathbb{Z}_p)^\times \\ b \in (\mathbb{Z}_p/p\mathbb{Z}_p)^\times \\ x \in \mathbb{Z}_p/p^4\mathbb{Z}_p}} \chi(b) \pi \left(\begin{bmatrix} 1 & -bp^{-1} & ap^{-2} \\ & 1 & ap^{-2} \\ & & 1 \\ & & & 1 \end{bmatrix} \right) v.
\end{aligned}$$

Proof. This result follows directly from Theorem 4.2. Converting to \mathbb{Q}_p and conjugating by C gives:

$$\begin{aligned}
T_\chi^1(v) &= p^2 \int_{\mathbb{Z}_p} \int_{\mathbb{Z}_p^\times} \int_{\mathbb{Z}_p^\times} \int_{\mathbb{Z}_p^\times} \chi(ab) \pi \left(C \begin{bmatrix} 1 & -(a+xb)p^{-1} & & \\ & 1 & & \\ & & 1 & (a+xb)p^{-1} \\ & & & 1 \end{bmatrix} \right. \\
&\quad \left. \begin{bmatrix} 1 & & bp^{-2} & zp^{-4} \\ & 1 & x^{-1}p^{-1} & bp^{-2} \\ & & 1 & \\ & & & 1 \end{bmatrix} C \right) v \, da \, db \, dx \, dz \\
&= p^2 \int_{\mathbb{Z}_p} \int_{\mathbb{Z}_p^\times} \int_{\mathbb{Z}_p^\times} \int_{\mathbb{Z}_p^\times} \chi(ab) \pi \left(\begin{bmatrix} 1 & -(a+xb)p^{-1} & & \\ & 1 & & \\ & & 1 & \\ & & & (a+xb)p^{-1} & 1 \end{bmatrix} \right. \\
&\quad \left. \begin{bmatrix} 1 & zp^{-4} & bp^{-2} \\ & 1 & bp^{-2} & x^{-1}p^{-1} \\ & & 1 & \\ & & & 1 \end{bmatrix} \right) v \, da \, db \, dx \, dz \\
&= p^{-11} \sum_{z \in \mathbb{Z}_p/p^4\mathbb{Z}_p} \sum_{a,b,x \in (\mathbb{Z}_p/p^3\mathbb{Z}_p)^\times} \chi(ab) \pi \left(\begin{bmatrix} 1 & -(a+xb)p^{-1} & & \\ & 1 & & \\ & & 1 & \\ & & & (a+xb)p^{-1} & 1 \end{bmatrix} \right. \\
&\quad \left. \begin{bmatrix} 1 & zp^{-4} & bp^{-2} \\ & 1 & bp^{-2} & x^{-1}p^{-1} \\ & & 1 & \\ & & & 1 \end{bmatrix} \right) v.
\end{aligned}$$

The remaining terms are similarly computed to complete the specialization to \mathbb{Q}_p . \square

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